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Phase-Space Representation of Quantum Systems in the Relative-State Formulation

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A phase-space representation of quantum systems within the framework of the relative-state formulation is proposed. To this end, relative-position and relative-momentum states are introduced and their properties are investigated in detail. Phase-space functions that represent a quantum state vector are constructed in terms of the relative-position and relative-momentum states, and the quantum dynamics is investigated by using the phase-space functions. Furthermore, probability distributions in phase space are considered by means of the relative-state formulation, and it is shown that the phase-space probability distribution is closely related to the operational probability distribution. The marginal distribution, characteristic function, and operational uncertainty relation are also discussed.

1. INTRODUCTION

Several authors have recently proposed phase-space representations of quantum systems by means of the abstract Hilbert space in order to investigate the time evolution of quantum systems in phase space and the correspondence principle between quantum and classical mechanics (Torres-Vega and Frederick, 1980, 1993; Harriman, 1994; Wlodarz, 1994). Conventionally, the Wigner function (Wigner, 1932; Hillery *et al.*, 1984) and Husimi function (Husimi, 1940; Kano, 1965; Mohta and Sudarshan, 1965) are used for these purposes. The phase-space representation is also useful in the group-theoretic approach to quantum mechanics (Kim and Noz, 1991). For example, a phase-space function that represents a quantum state of a single-particle system with one degree of freedom is given by a square-integrable function $\psi(r, k)$ which satisfies the normalization condition given by $\int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\, |\psi(r, k)|^2 = 1$. Obtaining expressions for the canonical position and momentum operators

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acting on the phase-space function (Torres-Vega and Frederick, 1980, 1993) and finding the kernel of the transformation between the phase space and position space (or momentum space) (Harriman, 1994) are key points of the phase-space approaches.

I have recently presented a relative-state formulation to describe a quantum system (Ban, 1991a, 1993a) and have shown (Ban, 1992a,b, 1993b, 1994a) that a quantum mechanical phase operator in quantum optics can be defined in terms of the relative-number states, free of the well-known difficulties in the conventional theories (Susskind and Glogower, 1994; Carruthers and Nieto, 1964). The average values of phase quantities calculated by the proposed method are equal to those obtained by the Pegg–Barnett phase operator method (Pegg and Barnett, 1988, 1989; Barnett and Pegg, 1989). In the relative-state formulation, a reference system is introduced to describe the physical properties of a relevant system. Thus the total system that is considered consists of the relevant and reference systems. The point is that a state of the reference system is determined, depending on what we would like to know about the relevant physical system. In this sense, the reference system may be considered a measurement apparatus. In some cases, however, the reference system may be a fictitious one. It will be shown here that the Hilbert space for the total system is analogous to that used to construct the phase-space representation (Torres-Vega and Frederick, 1980, 1993; Harriman, 1994) and that the relative-state formulation gives the phase-space representation in a natural way.

Quasiprobability distributions in quantum optics are important tools for investigating nonclassical properties of light (Gardiner, 1991; Carmichael, 1993; Walls and Milburn, 1994). The Glauber–Sudarshan P-function (Glauber, 1963a,b; Sudarshan, 1963), the Wigner function (Wigner, 1932; Hillery *et al.*, 1984), and the Q-function (or the Husimi function) (Husimi, 1940; Kano, 1965; Mehta and Sudarshan, 1965) are used extensively, and the generalized P-function is also used to investigate quantum optical systems (Drummond and Gardiner, 1980). It will also be shown here that another kind of phase-space probability distribution can be introduced by means of the relative-state formulation. Probability distributions in the relative-state formulation are positive-definite and are expressed as convolutions of the P-function and Q-functions and of the two Wigner functions. It will be shown that the phase-space probability distributions are closely related to the operational probability distribution (Wódkiewicz, 1984, 1986, 1987; Royer, 1985; Burak and Wódkiewicz, 1992) and the fuzzy-space distribution functions (Prugovečki, 1976a,b, 1978; Twareque Ali and Prugovečki, 1977).

In this paper, it will be shown that the phase-space representation and phase-space probability distribution of a quantum system can be constructed by means of the relative-state formulation, and their properties will be investigated in detail. For this purpose, the relative-position and relative-momentum states are introduced as basic tools. These states are defined on an extended Hilbert space that is a tensor product of two Hilbert spaces: the Hilbert space of the relevant physical system, and the Hilbert space of the reference system. The relative-position state describes the total system in terms of a certain reference position and a position relative to it. The relative-momentum state has the same meaning as the relative-position state, but in momentum space. The relative-position and relative-momentum states lead to the phase-space representation of quantum system. Furthermore, the phase-space probability distributions can be introduced in terms of the relative-position and relative-momentum states.

This paper is organized as follows. Section 2 introduces the relative-position and relative-momentum states in the extended Hilbert space and investigates their properties in detail. The expressions of position, momentum, annihilation, and creation operators acting on the relative-position and relative-momentum states are found. Section 3 obtains the reduced description of the relevant physical system by projecting the extended Hilbert space into the appropriate subspace. The reduced description leads to the phase-space representation of the relevant system. The quantum dynamics of pure and mixed states of the relevant system in the phase space are investigated. As examples, a free particle and a harmonic oscillator in the phase space are considered. Section 4 introduces probability distributions in the phase space by means of the relative-position and relative-momentum states and investigates their properties in detail. It is shown that these probability distributions are closely related to the operational probability distribution. Furthermore, the marginal distributions, characteristic function, and operational uncertainty relation are investigated, and several examples of the phase-space probability distributions are obtained. Section 5 summarizes the paper.

2. RELATIVE-POSITION AND RELATIVE-MOMENTUM STATES

2.1. Relative-Position State

This section introduces the relative-position and relative-momentum states and derives the expressions for position, momentum, annihilation, and creation operators acting on these states. In the relative-state formulation, we consider a composite system that consists of two independent systems: one is the relevant physical system and the other is a reference system. For simplicity here, we denote a state vector and an operator of the relevant system as $|\psi; A\rangle$ and $\hat{O}_A$, for example, and those of the reference system as $|\psi; B\rangle$ and $\hat{O}_B$. A state vector $|\psi; A\rangle$ of the relevant system is an element of a Hilbert space $\mathcal{H}_A$, and a state vector $|\psi; B\rangle$ of the reference system is an element of a Hilbert space $\mathcal{H}_B$. A state vector $|\Psi\rangle\rangle$ of the composite system that we consider is thus given by $|\Psi\rangle\rangle = |\psi; A\rangle \otimes |\phi; B\rangle$ and is an element of a tensor product space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. When we consider a mixed state, a statistical operator $\hat{W}$ of the composite system is given by $\hat{W} = \hat{\rho}_A \otimes \hat{\rho}_B$, where $\hat{\rho}_A$ and $\hat{\rho}_B$ are, respectively, statistical operators of the relevant and reference systems.

Let us first consider the relative-position state of the composite system. A set of position eigenstates becomes a complete orthonormal set. Denote position operators of the relevant and reference systems as $\hat{x}_A$ and $\hat{x}_B$ and their eigenstates as $|x; A\rangle$ and $|x; B\rangle$, such that $\hat{x}_A|x; A\rangle = x|x; A\rangle$ and $\hat{x}_B|x; B\rangle = x|x; B\rangle$. We assume, for simplicity, that each system has one degree of freedom. Then we have complete orthonormal sets of the relevant and reference systems, $\{|x; A\rangle | x \in \mathbf{R}\}$ and $\{|x; B\rangle | x \in \mathbf{R}\}$. Here $\mathbf{R}$ stands for the set of real numbers, and $|x; A\rangle$ and $|x; B\rangle$ satisfy the following orthogonality and completeness relations:

$$\langle \mathcal{S}; x | y; \mathcal{S}\rangle = \delta(x - y) \tag{2.1a}$$

$$\int_{-\infty}^{\infty} dx\, |x; \mathcal{S}\rangle\langle \mathcal{S}; x| = \hat{1}_{\mathcal{S}} \tag{2.1b}$$

where $\mathcal{S}$ stands for A or B, and where $\hat{1}_A$ and $\hat{1}_B$ are unit operators in the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. Thus a complete orthonormal set of the composite system is given by

$$S_{A+B} = \{|x, y\rangle\rangle = |x; A\rangle \otimes |y; B\rangle | x, y \in \mathbf{R}\} \tag{2.2}$$

where the vector $|x, y\rangle\rangle$ satisfies the relations

$$\langle\langle y, x | x', y'\rangle\rangle = \delta(x - x')\delta(y - y') \tag{2.3a}$$

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, |x, y\rangle\rangle\langle\langle y, x| = \hat{1}_A \otimes \hat{1}_B \tag{2.3b}$$

with $\langle\langle y, x| = (|x, y\rangle\rangle)^\dagger$.

The set S_{A+B} describes the composite system in terms of the positions of the relevant and reference systems. We would instead like to describe the composite system in terms of a certain reference position and a relative position between the relevant and reference systems. To this end, we introduce a state vector $|\pi_s(r, x)\rangle\rangle$ of the composite system:

$$|\pi_s(r, x)\rangle\rangle = |x + \tfrac{1}{2}(1 + s)r; A\rangle \otimes |x - \tfrac{1}{2}(1 - s)r; B\rangle \tag{2.4}$$

which satisfies the relations

$$(\hat{x}_A - \hat{x}_B)|\pi_s(r, x)\rangle\rangle = r|\pi_s(r, x)\rangle\rangle \tag{2.5a}$$

$$\tfrac{1}{2}(\hat{x}_A + \hat{x}_B)|\pi_s(r, x)\rangle\rangle = (x + \tfrac{1}{2}sr)|\pi_s(r, x)\rangle\rangle \tag{2.5b}$$

The relation

$$|x; A\rangle \otimes |y; B\rangle = \left|\pi_s\left(x - y, \frac{1-s}{2}x + \frac{1+s}{2}y\right)\right\rangle\!\!\Bigg\rangle$$

is also obtained. It is easy to see that the parameter r in the state $|\pi_s(r, x)\rangle\rangle$ represents the relative position between the relevant and reference systems. Thus, we refer to the state $|\pi_s(r, k)\rangle\rangle$ as the relative-position state. Since we obtain the relations

$$\hat{x}_A|\pi_s(r, x)\rangle\rangle = [x + \tfrac{1}{2}(1 + s)r]|\pi_s(r, x)\rangle\rangle \tag{2.6a}$$

$$\hat{x}_B|\pi_s(r, x)\rangle\rangle = [x - \tfrac{1}{2}(1 - s)r]|\pi_s(r, x)\rangle\rangle \tag{2.6b}$$

we see that if we set $s = 1, 0,$ or -1, the parameter x in the state $|\pi_s(r, x)\rangle\rangle$ respectively represents the position of the reference system, the center between the relevant and reference systems, or the position of the relevant system. A set of the relative-position states $\{|\pi_s(r, x)\rangle\rangle, |r, x \in \mathbf{R}\}$ becomes a complete orthonormal basis of the composite system, satisfying the relations

$$\langle\langle\pi_s(r, x)|\pi_s(r', x')\rangle\rangle = \delta(r - r')\delta(x - x') \tag{2.7a}$$

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dr\, |\pi_s(r, x)\rangle\rangle\langle\langle\pi_s(r, x)| = \hat{1}_A \otimes \hat{1}_B \tag{2.7b}$$

The state vector given by equation (2.4) is a generalization of the state vector introduced in Ban (1993a).

Next we introduce a state by Fourier transformation of the relative-position states $|\pi_s(r, x)\rangle\rangle$ with respect to the parameter x:

$$|\pi_s(r, k)\rangle\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, |\pi_s(r, x)\rangle\rangle e^{ikx} \tag{2.8}$$

This state plays an essential role in introducing the phase-space representation of quantum systems in Section 3. Since the Fourier transformation is unitary, the set of the transformed states $\{|\pi_s(r, k)\rangle\rangle | r, k \in \mathbf{R}\}$ becomes a complete orthonormal basis of the composite system and satisfies the relations

$$\langle\langle\pi_s(r, k)|\pi_s(r', k')\rangle\rangle = \delta(r - r')\delta(k - k') \tag{2.9a}$$

$$\int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\, |\pi_s(r, k)\rangle\rangle\langle\langle\pi_s(r, k)| = \hat{1}_A \otimes \hat{1}_B \tag{2.9b}$$

It is important to note that the state vector $|\pi_s(r, k)\rangle\rangle$ is a simultaneous eigenstate of the relative-position operator $\hat{x}_A - \hat{x}_B$ and the sum of the momentum operators $\hat{p}_A + \hat{p}_B$ (Helstrom, 1974; Leonhardt and Paul, 1993), where $\hat{p}_A$ and $\hat{p}_B$ are momentum operators of the relevant and reference systems and satisfy the canonical commutation relations $[\hat{x}_A, \hat{p}_A] = i\hat{1}_A$ and $[\hat{x}_B, \hat{p}_B] = i\hat{1}_B$. It is easy to verify the following eigenvalue equations:

$$(\hat{x}_A - \hat{x}_B)|\pi_s(r, k)\rangle\rangle = r|\pi_s(r, k)\rangle\rangle \tag{2.10a}$$

$$(\hat{p}_A + \hat{p}_B)|\pi_s(r, k)\rangle\rangle = k|\pi_s(r, k)\rangle\rangle \tag{2.10b}$$

We also refer to the state vector $|\pi_s(r, k)\rangle\rangle$ as the relative-position state. It is seen from equations (2.4) and (2.8) that the dependence of the relative-position state $|\pi_s(r, k)\rangle\rangle$ on the parameter s appears only in a phase factor:

$$|\pi_s(r, k)\rangle\rangle = |\pi(r, k)\rangle\rangle e^{i(1-s)kr/2} \tag{2.11}$$

where the state vector $|\pi(r, k)\rangle\rangle$ is given by

$$|\pi(r, k)\rangle\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, |x + r; A\rangle \otimes |x; B\rangle e^{ikx} \tag{2.12}$$

We introduce the following annihilation and creation operators of the relevant and reference systems:

$$\hat{a} = \frac{\hat{x}_A + i\hat{p}_A}{\sqrt{2}}, \qquad \hat{a}^\dagger = \frac{\hat{x}_A - i\hat{p}_A}{\sqrt{2}} \tag{2.13a}$$

$$\hat{b} = \frac{\hat{x}_B + i\hat{p}_B}{\sqrt{2}}, \qquad \hat{b}^\dagger = \frac{\hat{x}_B - i\hat{p}_B}{\sqrt{2}} \tag{2.13b}$$

which satisfy the commutation relations $[\hat{a}, \hat{a}^\dagger] = \hat{1}_A$ and $[\hat{b}, \hat{b}^\dagger] = \hat{1}_B$. We can then obtain the Fock representation of the relative-position state $|\pi_s(r, k)\rangle\rangle$:

$$\begin{aligned} |\pi_s(r, k)\rangle\rangle = {} & \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}|\mu|^2 + \mu\hat{a}^\dagger - \mu^*\hat{b}^\dagger + \hat{a}^\dagger\hat{b}^\dagger - \frac{1}{2}iskr\right) \\ & \times |0; A\rangle \otimes |0; B\rangle \end{aligned} \tag{2.14}$$

where $|0; A\rangle$ and $|0; B\rangle$ are vacuum states of the relevant and reference systems defined by the relations $\hat{a}|0; A\rangle = 0$ and $\hat{b}|0; B\rangle = 0$ and where the complex parameter μ is given by

$$\mu = \frac{r + ik}{\sqrt{2}} \tag{2.15}$$

In the definitions (2.13), we have assumed the position and momentum to be dimensionless. The derivation of the Fock representation (2.14) is given in the Appendix.

2.2. Relative-Momentum State

The relative-momentum state $|\tilde{\pi}_s(p, k)\rangle\rangle$ of the composite system is defined as

$$|\tilde{\pi}_s(p, k)\rangle\rangle = |p + \tfrac{1}{2}(1 + s)k; A\rangle \otimes |p - \tfrac{1}{2}(1 - s)k; B\rangle \tag{2.16}$$

where $|p; A\rangle$ and $|p; B\rangle$ are eigenstates of the canonical momentum operators $\hat{p}_A$ and $\hat{p}_B$ of the relevant and reference systems: $\hat{p}_A|p; A\rangle = p|p; A\rangle$ and $\hat{p}_B|p; B\rangle = p|p; B\rangle$. The relative-position state $|\tilde{\pi}_s(p, k)\rangle\rangle$ satisfies the relations

$$(\hat{p}_A - \hat{p}_B)|\tilde{\pi}_s(p, k)\rangle\rangle = k|\tilde{\pi}_s(p, k)\rangle\rangle \tag{2.17a}$$

$$\tfrac{1}{2}(\hat{p}_A + \hat{p}_B)|\tilde{\pi}_s(p, k)\rangle\rangle = (p + \tfrac{1}{2}sk)|\tilde{\pi}_s(p, k)\rangle\rangle \tag{2.17b}$$

The relation

$$|p; A\rangle \otimes |p'; B\rangle = \left|\tilde{\pi}_s\left(p - p', \frac{1-s}{2}p + \frac{1+s}{2}p'\right)\right\rangle\!\!\right\rangle$$

is also obtained. It is easy to see that the parameter k in the state $|\tilde{\pi}_s(p, k)\rangle\rangle$ indicates the relative momentum between the relevant and reference systems. It is also seen that if we set $s = 1$, 0, or -1, the parameter p in the state $|\tilde{\pi}_s(p, k)\rangle\rangle$ respectively represents the momentum of the reference system, the average momentum of the relevant and reference systems, or the momentum of the relevant system. Because sets $\{|p; A\rangle | p \in \mathbf{R}\}$ and $\{|p; B\rangle | p \in \mathbf{R}\}$ are complete orthonormal bases of the relevant and reference systems, respectively, it is found that $\{|\tilde{\pi}_s(p; k)\rangle\rangle | p, k \in \mathbf{R}\}$ becomes a complete orthonormal basis of the composite system, satisfying the relations

$$\langle\langle\tilde{\pi}_s(p, k)|\tilde{\pi}_s(p', k')\rangle\rangle = \delta(k - k')\delta(p - p') \tag{2.18a}$$

$$\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dp\, |\tilde{\pi}_s(p, k)\rangle\rangle\langle\langle\tilde{\pi}_s(p, k)| = \hat{1}_A \otimes \hat{1}_B \tag{2.18b}$$

The description of the composite system in terms of the relative-momentum states $|\tilde{\pi}_s(p, k)\rangle\rangle$ in momentum space is the same as that in terms of the relative-position states $|\pi_s(r, x)\rangle\rangle$ in position space.

Next we introduce another relative-momentum state $|\tilde{\pi}_s(r, k)\rangle\rangle$ by Fourier transformation of $|\tilde{\pi}_s(p, k)\rangle\rangle$ with respect to p:

$$|\tilde{\pi}_s(r, k)\rangle\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp\, |\tilde{\pi}_s(p, k)\rangle\rangle e^{-ipr} \tag{2.19}$$

The relative-momentum state $|\tilde{\pi}_s(r, k)\rangle\rangle$ is a simultaneous eigenstate of the difference of the momentum operators $\hat{p}_A - \hat{p}_B$ and the sum of the position operators $\hat{x}_A + \hat{x}_B$,

$$(\hat{p}_A - \hat{p}_B)|\tilde{\pi}_s(r, k)\rangle\rangle = k|\tilde{\pi}_s(r, k)\rangle\rangle \tag{2.20a}$$

$$(\hat{x}_A + \hat{x}_B)|\tilde{\pi}_s(r, k)\rangle\rangle = r|\tilde{\pi}_s(r, k)\rangle\rangle \tag{2.20b}$$

It is easy to see that the set $\{|\tilde{\pi}_s(r, k)\rangle\rangle | r, k \in \mathbf{R}\}$ becomes a complete orthonormal basis of the composite system and satisfies the relations

$$\langle\langle\tilde{\pi}_s(r, k)|\tilde{\pi}_s(r', k')\rangle\rangle = \delta(r - r')\delta(k - k') \tag{2.21a}$$

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dk\, |\tilde{\pi}_s(r, k)\rangle\rangle\langle\langle\tilde{\pi}_s(r, k)| = \hat{1}_A \otimes \hat{1}_B \tag{2.21b}$$

The dependence of the relative-momentum state $|\tilde{\pi}_s(r, k)\rangle\rangle$ on the parameter s appears only in a phase factor:

$$|\tilde{\pi}_s(r, k)\rangle\rangle = |\tilde{\pi}(r, k)\rangle\rangle e^{-i(1-s)kr/2} \tag{2.22}$$

where the state vector $|\tilde{\pi}(r, k)\rangle\rangle$ is given by

$$|\tilde{\pi}(r, k)\rangle\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp\, |p + k; A\rangle \otimes |p; B\rangle e^{-ipr} \tag{2.23}$$

The Fock representation of the relative-momentum state $|\tilde{\pi}(r, k)\rangle\rangle$ is given by

$$\begin{aligned} |\tilde{\pi}_s(r, k)\rangle\rangle = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}|\mu|^2 + \mu\hat{a}^\dagger + \mu^*\hat{b}^\dagger - \hat{a}^\dagger\hat{b}^\dagger + \frac{1}{2} iskr\right) \\ \times |0; A\rangle \otimes |0; B\rangle \end{aligned} \tag{2.24}$$

where the complex parameter μ is given by equation (2.15). The derivation of this expression is given in the Appendix.

2.3. Expressions of Basic Operators

Finally, we obtain the expressions for the position, momentum, annihilation, and creation operators acting on the relative-position state $|\pi_s(r, k)\rangle\rangle$ and relative-momentum state $|\tilde{\pi}_s(r, k)\rangle\rangle$. To this end, let us suppose that the composite system is described by a state vector $|\Psi\rangle\rangle$ which is an element of the extended Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, and let us set

$$\Psi_s(r, k) = \langle\langle \pi_s(r, k) | \Psi \rangle\rangle \tag{2.25a}$$

$$\tilde{\Psi}_s(r, k) = \langle\langle \tilde{\pi}_s(r, k) | \Psi \rangle\rangle \tag{2.25b}$$

which are square-integrable functions satisfying the normalization conditions

$$\int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\, |\Psi_s(r, k)|^2 = \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\, |\tilde{\Psi}_s(r, k)|^2 = 1 \tag{2.26}$$

First consider the function $\Psi_s(r, k)$ in terms of the relative-position states. Using equations (2.4) and (2.8), we obtain the expressions for the canonical position and momentum operators $(\hat{x}_A, \hat{p}_A)$ and $(\hat{x}_B, \hat{p}_B)$:

$$\langle\langle \pi_s(r, k) | \hat{x}_A | \Psi \rangle\rangle = [\tfrac{1}{2}(1 + s)r + i\partial_k]\Psi_s(r, k) \tag{2.27a}$$

$$\langle\langle \pi_s(r, k) | \hat{p}_A | \Psi \rangle\rangle = [\tfrac{1}{2}(1 - s)k - i\partial_r]\Psi_s(r, k) \tag{2.27b}$$

and

$$\langle\langle \pi_s(r, k) | \hat{x}_B | \Psi \rangle\rangle = -[\tfrac{1}{2}(1 - s)r - i\partial_k]\Psi_s(r, k) \tag{2.28a}$$

$$\langle\langle \pi_s(r, k) | \hat{p}_B | \Psi \rangle\rangle = [\tfrac{1}{2}(1 + s)k + i\partial_r]\Psi_s(r, k) \tag{2.28b}$$

where ∂_r and ∂_k stand for $\partial/\partial r$ and $\partial/\partial k$, respectively. Furthermore, using the relations (2.13a) and (2.13b), we can obtain the expressions for the annihilation and creation operators $(\hat{a}, \hat{a}^\dagger)$ and $(\hat{b}, \hat{b}^\dagger)$ from equations (2.27) and (2.28):

$$\langle\langle \pi_s(r, k) | \hat{a} | \Psi \rangle\rangle = [\tfrac{1}{2}(\mu + s\mu^*) + \partial_{\mu^*}]\Psi_s(r, k) \tag{2.29a}$$

$$\langle\langle \pi_s(r, k) | \hat{a}^\dagger | \Psi \rangle\rangle = [\tfrac{1}{2}(\mu^* + s\mu) - \partial_{\mu}]\Psi_s(r, k) \tag{2.29b}$$

and

$$\langle\langle \pi_s(r, k) | \hat{b} | \Psi \rangle\rangle = -[\tfrac{1}{2}(\mu^* - s\mu) + \partial_{\mu}]\Psi_s(r, k) \tag{2.30a}$$

$$\langle\langle \pi_s(r, k) | \hat{b}^\dagger | \Psi \rangle\rangle = -[\tfrac{1}{2}(\mu - s\mu^*) - \partial_{\mu^*}]\Psi_s(r, k) \tag{2.30b}$$

where the complex parameter μ is given by equation (2.15) and we set $\partial_\mu = \partial/\partial\mu$ and $\partial_{\mu^*} = \partial/\partial\mu^*$.

Now consider the function $\tilde{\Psi}_s(r, k)$ in terms of the relative-momentum states. When we use equations (2.16) and (2.19), we obtain the expressions for the position and momentum operators:

$$\langle\langle \tilde{\pi}_s(r, k) | \hat{x}_A | \Psi \rangle\rangle = [\tfrac{1}{2}(1 - s)r + i\partial_k]\tilde{\Psi}_s(r, k) \tag{2.31a}$$

$$\langle\langle \tilde{\pi}_s(r, k) | \hat{p}_A | \Psi \rangle\rangle = [\tfrac{1}{2}(1 + s)k - i\partial_r]\tilde{\Psi}_s(r, k) \tag{2.31b}$$

and

$$\langle\langle\tilde{\pi}_s(r, k)|\hat{x}_B|\Psi\rangle\rangle = [\tfrac{1}{2}(1 + s)r - i\partial_k]\tilde{\Psi}_s(r, k) \tag{2.32a}$$

$$\langle\langle\tilde{\pi}_s(r, k)|\hat{p}_B|\Psi\rangle\rangle = -[\tfrac{1}{2}(1 - s)k + i\partial_r]\tilde{\Psi}_s(r, k) \tag{2.32b}$$

For annihilation and creation operators, we can get the following relations:

$$\langle\langle\tilde{\pi}_s(r, k)|\hat{a}|\Psi\rangle\rangle = [\tfrac{1}{2}(\mu - s\mu^*) + \partial_{\mu^*}]\tilde{\Psi}_s(r, k) \tag{2.33a}$$

$$\langle\langle\tilde{\pi}_s(r, k)|\hat{a}^\dagger|\Psi\rangle\rangle = [\tfrac{1}{2}(\mu^* - s\mu) - \partial_{\mu}]\tilde{\Psi}_s(r, k) \tag{2.33b}$$

and

$$\langle\langle\tilde{\pi}_s(r, k)|\hat{b}|\Psi\rangle\rangle = [\tfrac{1}{2}(\mu^* + s\mu) + \partial_{\mu}]\tilde{\Psi}_s(r, k) \tag{2.34a}$$

$$\langle\langle\tilde{\pi}_s(r, k)|\hat{b}^\dagger|\Psi\rangle\rangle = [\tfrac{1}{2}(\mu + s\mu^*) - \partial_{\mu^*}]\tilde{\Psi}_s(r, k) \tag{2.34b}$$

The expressions (2.27)–(2.30) and (2.31)–(2.34) of the basic operators are used to investigate the dynamics of a quantum system in the phase space.

3. PHASE-SPACE REPRESENTATION OF QUANTUM STATE

3.1. Reduced Description of the Relevant System

To describe the composite system, we have introduced the relative-position and relative-momentum states $|\pi_s(r, k)\rangle\rangle$ and $|\tilde{\pi}_s(r, k)\rangle\rangle$. When we wish to investigate only the relevant system, however, the extended Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ to which the state vectors $|\pi_s(r, k)\rangle\rangle$ and $|\tilde{\pi}_s(r, k)\rangle\rangle$ belong is too large. Thus, to describe the quantum mechanical properties of the relevant system, we have to restrict the extended Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ to an appropriate subspace $\mathcal{H}_0$. To this end, let us assume that we know the state of the reference system and that the state of the reference system remains unchanged (Stenholm, 1980). We denote this state as $|\phi; B\rangle$. This indicates the restriction of the extended Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ to the subspace $\mathcal{H}_0 = \mathcal{H}_A \otimes \{|\phi; B\rangle\}$. It should be noted here that we never trace out the reference system. Thus in this reduced description we can consider the following states as the relative-position and relative-momentum states of the relevant system:

$$\begin{aligned}|\pi_s(r, k); A\rangle &\equiv \langle B; \phi|\pi_s(r, k)\rangle\rangle \\ &= \frac{1}{\sqrt{2\pi}} e^{-i(1+s)kr/2} \int_{-\infty}^{\infty} dx\, |x; A\rangle\phi^*(x - r)e^{ikx}\end{aligned} \tag{3.1}$$

and

$$|\tilde{\pi}_s(r, k); A\rangle \equiv \langle B; \phi | \tilde{\pi}_s(r, k)\rangle\rangle$$

$$= \frac{1}{\sqrt{2\pi}} e^{i(1+s)kr/2} \int_{-\infty}^{\infty} dp \, |p; A\rangle \tilde{\phi}^*(p - k) e^{-ipr} \quad (3.2)$$

where we set $\phi(x) = \langle B; x | \phi; B\rangle$ and $\tilde{\phi}(p) = \langle B; p | \phi; B\rangle$.

Throughout Section 3 we ignore the index A of a state vector of the relevant system unless that would introduce confusion. Thus the state vector written without an index explicitly belongs to the Hilbert space $\mathcal{H}_A$. The states $|\pi_s(r, k)\rangle\rangle$ and $|\tilde{\pi}_s(r, k)\rangle\rangle$, for example, indicate $|\pi_s(r, k); A\rangle$ and $|\tilde{\pi}_s(r, k); A\rangle$. In the following, we will refer to the state given by equation (3.1) as the reduced relative-position state and to that given by (3.2) as the reduced relative-momentum state.

Let us first investigate the properties of the reduced relative-position state $|\pi_s(r, k)\rangle\rangle$. It is easy to see that a set of the reduced relative-position states, $\{|\pi_s(r, k)\rangle\rangle | r, k \in \mathbf{R}\}$, becomes a complete nonorthogonal set of the relevant system and satisfies the relations

$$\int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk \, |\pi_s(r, k)\rangle\rangle\langle\langle\pi_s(r, k)|$$

$$= \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dx \, |x\rangle\langle x| \, |\phi(x - r)|^2 = \hat{1} \quad (3.3a)$$

$$\langle\langle\pi_s(r, k) | \pi_s(r', k')\rangle\rangle$$

$$= \frac{1}{2\pi} e^{i(1+s)(kr - k'r')/2} \int_{-\infty}^{\infty} dx \, \phi(x - r)\phi^*(x - r') e^{-i(k-k')x} \quad (3.3b)$$

When we set $r = r'$ and $k = k'$ in equation (3.3b), we obtain the normalization condition of the reduced relative-position state: $\langle\langle\pi_s(r, k) | \pi_s(r, k)\rangle\rangle = 1$. The nonorthogonality of the reduced relative-position state is due to the restriction of the Hilbert space. The completeness relation (3.3a) ensures that any state vector of the relevant system can be expanded in terms of the reduced relative-position states. When the reference system is in a coherent state $|\beta; B\rangle$, such that $\hat{b}|\beta; B\rangle = \beta|\beta; B\rangle$, the reduced relative-position state $|\pi_s(r, k)\rangle\rangle$ becomes

$$|\pi_s(r, k; \beta)\rangle\rangle = \frac{1}{\sqrt{2\pi}} e^{i[\mathrm{Im}(\beta\mu) - skr/2]} |\mu + \beta^*\rangle \quad (3.4)$$

where $|\mu\rangle$ and $|0\rangle$ are respectively the coherent and vacuum states of the relevant system and where the complex amplitude μ is given by $\mu = (r + ik)/\sqrt{2}$. In deriving equation (3.4), we have used the Fock representation of the relative-position state given by equation (2.14). The state $|\pi_s(r, k; \beta)\rangle$ is nothing but a coherent state, and in this case equations (3.3a) and (3.3b) become

$$\int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\, |\pi_s(r, k)\rangle\langle\pi_s(r, k)|$$

$$= \frac{1}{\pi}\int_{\mathbf{R}^2} d^2\mu\, |\mu\rangle\langle\mu| = \hat{1} \tag{3.5a}$$

$$\langle\pi_s(r, k)|\pi_s(r', k')\rangle$$

$$= \frac{1}{2\pi}\exp\left[-\frac{1}{2}(|\mu + \beta^*|^2 + |\mu' + \beta^*|^2) - (\mu^* + \beta)(\mu' + \beta^*) + \frac{1}{2}is(kr - k'r')\right] \tag{3.5b}$$

In particular, when the reference system is in the vacuum state $|\phi; B\rangle = |0; B\rangle$, we obtain

$$|\pi_s(r, k; 0)\rangle = \frac{1}{\sqrt{2\pi}} e^{-iskr/2}|\mu\rangle \tag{3.6}$$

For a wave packet state $|\psi\rangle$ of the relevant system, we can use equations (2.27) and (2.29) to obtain the following expressions for the position, momentum, annihilation, and creation operators of the relevant system:

$$\langle\pi_s(r, k)|\hat{x}|\psi\rangle = [\tfrac{1}{2}(1 + s)r + i\partial_k]\psi_s(r, k) \tag{3.7a}$$

$$\langle\pi_s(r, k)|\hat{p}|\psi\rangle = [\tfrac{1}{2}(1 - s)k - i\partial_r]\psi_s(r, k) \tag{3.7b}$$

and

$$\langle\pi_s(r, k)|\hat{a}|\psi\rangle = [\tfrac{1}{2}(\mu + s\mu^*) + \partial_{\mu^*}]\psi_s(r, k) \tag{3.8a}$$

$$\langle\pi_s(r, k)|\hat{a}^\dagger|\psi\rangle = [\tfrac{1}{2}(\mu^* + s\mu) - \partial_{\mu}]\psi_s(r, k) \tag{3.8b}$$

where we set $\psi_s(r, k) = \langle\pi_s(r, k)|\psi\rangle$, which is a square-integrable function.

Now consider the properties of the reduced relative-momentum state $|\tilde{\pi}_s(r, k)\rangle$ given by equation (3.2). It is easy to see that a set of the reduced relative-momentum states given by $\{|\tilde{\pi}_s(r, k)\rangle | r, k \in \mathbf{R}\}$ becomes a complete nonorthogonal set which satisfies the relations

$$\int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\, |\tilde{\pi}_s(r, k)\rangle\langle\tilde{\pi}_s(r, k)|$$

$$= \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dp\, |p\rangle\langle p| \cdot |\tilde{\phi}(p - k)|^2 = \hat{1} \tag{3.9a}$$

$$\langle\tilde{\pi}_s(r, k)|\tilde{\pi}_s(r', k')\rangle$$

$$= \frac{1}{2\pi} e^{-i(1+s)(kr-k'r')/2} \int_{-\infty}^{\infty} dp\, \tilde{\phi}(p - k)\tilde{\phi}^*(p - k')e^{ip(r-r')} \tag{3.9b}$$

When we set $r = r'$ and $k = k'$ in equation (3.9b), we obtain the normalization condition of the reduced relative-momentum state: $\langle\tilde{\pi}_s(r, k)|\tilde{\pi}_s(r, k)\rangle = 1$. When the reference system is in a coherent state $|\phi; B\rangle = |\beta; B\rangle$, the reduced relative-momentum state becomes

$$|\tilde{\pi}_s(r, k; \beta)\rangle = \frac{1}{\sqrt{2\pi}} e^{-i[\mathrm{Im}(\mu\beta)-skr/2]} |\mu - \beta^*\rangle \tag{3.10}$$

where we have used the Fock representation of the relative-momentum state given by equation (2.24). In this case the reduced relative-momentum state is nothing but a coherent state. In particular, we obtain for a vacuum state of the reference system

$$|\tilde{\pi}_s(r, k; 0)\rangle = \frac{1}{\sqrt{2\pi}} e^{iskr/2} |\mu\rangle \tag{3.11}$$

For a wave packet state $|\psi\rangle$ of the relevant system, we can use equations (2.31) and (2.33) to obtain the expressions for the position, momentum, annihilation, and creation operators of the relevant system:

$$\langle\tilde{\pi}_s(r, k)|\hat{x}|\psi\rangle = [\tfrac{1}{2}(1 - s)r + i\partial_k]\tilde{\psi}_s(r, k) \tag{3.12a}$$

$$\langle\tilde{\pi}_s(r, k)|\hat{p}|\psi\rangle = [\tfrac{1}{2}(1 + s)k - i\partial_r]\tilde{\psi}_s(r, k) \tag{3.12b}$$

and

$$\langle\tilde{\pi}_s(r, k)|\hat{a}|\psi\rangle = [\tfrac{1}{2}(\mu - s\mu^*) + \partial_{\mu^*}]\tilde{\psi}_s(r, k) \tag{3.13a}$$

$$\langle\tilde{\pi}_s(r, k)|\hat{a}^\dagger|\psi\rangle = [\tfrac{1}{2}(\mu^* - s\mu) - \partial_{\mu}]\tilde{\psi}_s(r, k) \tag{3.13b}$$

where we set $\tilde{\psi}_s(r, k) = \langle\tilde{\pi}_s(r, k)|\psi\rangle$, which is a square-integrable function.

Before leaving this section, let us consider the relation between the reduced relative-position and reduced relative-momentum states. Since $\tilde{\phi}(p)$

is given by the Fourier transformation of $\phi(x)$, we can obtain from equation (3.2)

$$|\tilde{\pi}_s(r, k)\rangle = \frac{1}{\sqrt{2\pi}} e^{-i(1-s)kr/2} \int_{-\infty}^{\infty} dx\, |x\rangle \phi^*(r - x) e^{ikx} \tag{3.14}$$

Comparing this with equation (3.1), we obtain

$$|\pi_s(r, k)\rangle = |\tilde{\pi}_{-s}(r, k)\rangle \tag{3.15}$$

if $\phi(x)$ is an even function; that is, if $\phi(x) = \phi(-x)$. For example, a vacuum state of the reference system satisfies this condition because $\phi(x) = \pi^{-1/4} \exp(-x^2/2)$.

3.2. Phase-Space Function of Quantum State

Let us now consider a phase-space function which represents a quantum state of the relevant system. Suppose that the relevant system is described by a state vector $|\psi\rangle$. Since both the reduced relative-position and relative-momentum states $|\pi_s(r, k)\rangle$ and $|\tilde{\pi}_s(r, k)\rangle$ satisfy the completeness given by equations (3.3a) and (3.9a), we can express the state vector $|\psi\rangle$ of the relevant system as

$$\begin{aligned} |\psi\rangle &= \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\, \psi_s(r, k) |\pi_s(r, k)\rangle \\ &= \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\, \tilde{\psi}_s(r, k) |\tilde{\pi}_s(r, k)\rangle \end{aligned} \tag{3.16}$$

where the functions $\psi_s(r, k)$ and $\tilde{\psi}_s(r, k)$ are given by

$$\psi_s(r, k) = \langle \pi_s(r, k) | \psi\rangle \tag{3.17a}$$

$$\tilde{\psi}_s(r, k) = \langle \tilde{\pi}_s(r, k) | \psi\rangle \tag{3.17b}$$

The normalization condition of the state vector requires that $\psi_s(r, k)$ and $\tilde{\psi}_s(r, k)$ should satisfy the relations

$$\int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dr' \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' \times \psi_s^*(r, k) \psi_s(r', k') \langle \pi_s(r, k) | \pi_s(r', k')\rangle = 1 \tag{3.18a}$$

$$\int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dr' \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' \times \tilde{\psi}_s^*(r, k) \tilde{\psi}_s(r', k') \langle \tilde{\pi}_s(r, k) | \tilde{\pi}_s(r', k')\rangle = 1 \tag{3.18b}$$

Furthermore, equations (3.17a) and (3.17b) lead to the normalization conditions for $\psi_s(r, k)$ and $\tilde{\psi}_s(r, k)$:

$$\int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk \, |\psi_s(r, k)|^2 = \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk \, |\tilde{\psi}_s(r, k)|^2 = 1 \qquad (3.19)$$

When a state vector $|\psi\rangle$ is given, we get the functions $\psi_s(r, k)$ and $\tilde{\psi}_s(r, k)$ from equations (3.17a) and (3.17b). On the other hand, if functions $\psi_s(r, k)$ and $\tilde{\psi}_s(r, k)$ that satisfy the conditions (3.18) and (3.19) are given, we can obtain the normalized vector $|\psi\rangle$ through equation (3.16). Thus, we find that a quantum state of the relevant system can be represented by the square-integrable function $\psi_s(r, k)$ or $\tilde{\psi}_s(r, k)$. The phase-space probability distribution is given by $|\psi_s(r, k)|^2$ or $|\tilde{\psi}_s(r, k)|^2$. We will refer to $\psi_s(r, k)$ and $\tilde{\psi}_s(r, k)$ as the phase-space functions that represent a quantum state $|\psi\rangle$. In the following, the usual wave functions in position and momentum space will be denoted as $\psi(x) = \langle x|\psi\rangle$ and $\tilde{\psi}(p) = \langle p|\psi\rangle$, respectively.

First consider the phase-space function $\psi_s(r, k) = \langle \pi_s(r, k)|\psi\rangle$. It is easy to see that a phase-space function $\psi_s(r, k)$ and wave function $\psi(x)$ representing the same quantum state $|\psi\rangle$ are connected by the transformations

$$\psi_s(r, k) = \int_{-\infty}^{\infty} dx \, T_s(r, k|x)\psi(x) \qquad (3.20a)$$

$$\psi(x) = \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk \, T_s^*(r, k|x)\psi_s(r, k) \qquad (3.20b)$$

where the kernel $T_s(r, k|x)$ of the transformation between the phase space and position space is given by

$$\begin{aligned} T_s(r, k|x) &= \langle \pi_s(r, k)|x\rangle \\ &= \frac{1}{\sqrt{2\pi}} \phi(x - r)e^{-ik[x-(1+s)r/2]} \end{aligned} \qquad (3.21)$$

The transformation kernel satisfies the relations

$$\int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk \, T_s(r, k|x)T_s^*(r, k|x') = \delta(x - x') \qquad (3.22a)$$

$$\int_{-\infty}^{\infty} dx \, T_s(r, k|x)T_s^*(r', k'|x) = \langle \pi_s(r, k)|\pi_s(r', k')\rangle \qquad (3.22b)$$

where the right-hand side of equation (3.22b) is given by equation (3.3b).

Similarly, the relationship between the phase-space function $\psi_s(r, k)$ and the wave function $\tilde{\psi}(p)$ in momentum space is given by

$$\psi_s(r, k) = \int_{-\infty}^{\infty} dp\, T_s(r, k \mid p)\tilde{\psi}(p) \tag{3.23a}$$

$$\tilde{\psi}(p) = \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\, T_s^*(r, k \mid p)\psi_s(r, k) \tag{3.23b}$$

where the transformation kernel $T_s(r, k \mid p)$ is given by

$$\begin{aligned} T_s(r, k \mid p) &= \langle \pi_s(r, k) \mid p \rangle \\ &= \frac{1}{\sqrt{2\pi}}\, \tilde{\phi}(k - p) e^{i[p-(1-s)k/2]r} \end{aligned} \tag{3.24}$$

which satisfies relations analogous to equations (3.22a) and (3.22b) but with p and p' replacing x and x'. In particular, if the reference system is in the vacuum state $|\phi; B\rangle = |0; B\rangle$, the transformation kernels $T_s(r, k \mid x)$ and $T_s(r, k \mid p)$ become Gaussian functions:

$$T_s(r, k \mid x) = \frac{1}{(2\pi^{3/2})^{1/2}} \exp\left[-\frac{1}{2}(x - r)^2 - ikx + i\,\frac{1 + s}{2}\, kr \right] \tag{3.25a}$$

$$T_s(r, k \mid p) = \frac{1}{(2\pi^{3/2})^{1/2}} \exp\left[-\frac{1}{2}(p - k)^2 + ipr - i\,\frac{1 - s}{2}\, kr \right] \tag{3.25b}$$

If we substitute $s = 0$ into these functions, we obtain the transformation kernels equivalent to those obtained in Harriman (1994). In this case, the kernels $T_s(r, k \mid x)$ and $T_s(r, k \mid p)$ lead to the Husimi transformations.

Now consider the phase-space function $\tilde{\psi}_s(r, k) = \langle \tilde{\pi}_s(r, k) \mid \psi \rangle$. When we use the completeness relation of the reduced relative-momentum state given by (3.9a), we obtain the kernel $\tilde{T}_s(r, k \mid x)$ of the transformation between the phase space and position space,

$$\begin{aligned} \tilde{T}_s(r, k \mid x) &= \langle \tilde{\pi}_s(r, k) \mid x \rangle \\ &= \frac{1}{\sqrt{2\pi}}\, \phi(r - x) e^{-ik[x-(1-s)r/2]} \end{aligned} \tag{3.26}$$

which leads to the transformation between $\tilde{\psi}_s(r, k)$ and $\psi(x)$:

$$\tilde{\psi}_s(r, k) = \int_{-\infty}^{\infty} dx\, \tilde{T}_s(r, k \mid x)\psi(x) \tag{3.27a}$$

$$\psi(x) = \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\, \tilde{T}_s^*(r, k \mid x)\tilde{\psi}_s(r, k) \tag{3.27b}$$

We further obtain the kernel $\tilde{T}_s(r, k|p)$ of the transformation between the phase space and momentum space:

$$\tilde{T}_s(r, k|p) = \langle \tilde{\pi}_s(r, k)|p\rangle = \frac{1}{\sqrt{2\pi}} \tilde{\phi}(p - k)e^{i[p-(1+s)k/2]r} \tag{3.28}$$

Thus we obtain the transformation between $\tilde{\psi}_s(r, k)$ and $\tilde{\psi}(p)$:

$$\tilde{\psi}_s(r, k) = \int_{-\infty}^{\infty} dp\ \tilde{T}_s(r, k|p)\tilde{\psi}(p) \tag{3.29a}$$

$$\tilde{\psi}(p) = \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\ \tilde{T}_s^*(r, k|p)\tilde{\psi}_s(r, k) \tag{3.29b}$$

The transformation kernels $\tilde{T}_s(r, k|x)$ and $\tilde{T}_s(r, k|p)$ satisfy the relations

$$\int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\ \tilde{T}_s(r, k|z)\tilde{T}_s^*(r, k|z') = \delta(z - z') \tag{3.30a}$$

$$\int_{-\infty}^{\infty} dz\ \tilde{T}_s(r, k|z)\tilde{T}_s^*(r', k'|z) = \langle \tilde{\pi}_s(r, k)|\tilde{\pi}_s(r', k')\rangle \tag{3.30b}$$

where the variable z stands for x or p. In particular, if the reference system is in the vacuum state, the transformation kernels $\tilde{T}_s(r, k|x)$ and $\tilde{T}_s(r, k|p)$ are given by

$$\tilde{T}_s(r, k|x) = \frac{1}{(2\pi^{3/2})^{1/2}} \exp\left[-\frac{1}{2}(x - r)^2 - ikx + i\frac{1 - s}{2}kr\right] \tag{3.31a}$$

$$\tilde{T}_s(r, k|p) = \frac{1}{(2\pi^{3/2})^{1/2}} \exp\left[-\frac{1}{2}(p - k)^2 + ipr - i\frac{1 + s}{2}kr\right] \tag{3.31b}$$

When we substitute $s = 0$ into these functions, the transformation kernels become equivalent to those obtained in Harriman (1994) and lead to the Husimi transformations.

Now consider the relation between the transformation kernels. Suppose that $\phi(x) = \langle B; x|\psi; B\rangle$ is an even function with respect to x; that is, $\phi(x) = \phi(-x)$. In this case, since we have the relation (3.15) between the reduced relative-position and relative-momentum states, we can obtain the following relations:

$$\tilde{T}_s(r, k|x) = T_{-s}(r, k|x) = e^{-iskr}T_s(r, k|x) \tag{3.32a}$$

$$\tilde{T}_s(r, k|p) = T_{-s}(r, k|p) = e^{-iskr}T_s(r, k|p) \tag{3.32b}$$

Note that the kernels given by equations (3.25) and (3.31) satisfy these relations because we have $\phi(x) = \pi^{-1/4} \exp(-x^2/2)$ for the vacuum state of the reference system.

3.3. Dynamics of Quantum System in the Phase Space

Now let us use the phase-space functions $\psi_s(r, k)$ and $W_s(r, k \mid r', k')$ to consider the quantum dynamics of the relevant system in the phase space. It should be noted here that the composite system consists of the relevant and reference systems. In this consideration, it is postulated that the relevant system evolves with time according to its own dynamics governed by a Hamiltonian and that the reference system is fixed in a certain state and does not evolve with time (Stenholm, 1980). It is also assumed here that the relevant system does not interact with the reference system through an interaction Hamiltonian.

First consider the case in which the system is described by a pure state. According to the above assumption, the state of the composite system at time t can be expressed as

$$|\Psi(t)\rangle\rangle = |\psi(t)\rangle \otimes |\phi; B\rangle \tag{3.33}$$

where $|\psi(t)\rangle$ is the state of the relevant system at time t and $|\phi; B\rangle$ is the fixed state of the reference system. The time evolution is governed by the Schrödinger equation

$$\partial_t |\Psi(t)\rangle\rangle = -i\hat{H} |\Psi(t)\rangle\rangle \tag{3.34}$$

where $\hat{H} = H[\hat{x}, \hat{p}]$ is the Hamiltonian of the relevant system and ∂_t stands for ∂/∂_t. Using the relations

$$\langle\langle \pi_s(r, k) | \Psi(t)\rangle\rangle = \langle \pi_s(r, k) | \psi(t)\rangle \equiv \psi_s(t; r, k) \tag{3.35a}$$

$$\langle\langle \pi_s(r, k) | H(\hat{x}, \hat{p}) | \Psi(t)\rangle\rangle = H[\tfrac{1}{2}(1 + s)r + i\partial_k, \tfrac{1}{2}(1 - s)k - i\partial_r]\psi_s(t; r, k) \tag{3.35b}$$

we obtain the equation of motion for the phase-space function $\psi_s(t; r, k)$ of the relevant system:

$$\partial_t \psi_s(t; r, k) = -iH[\tfrac{1}{2}(1 + s)r + i\partial_k, \tfrac{1}{2}(1 - s)k - i\partial_r]\psi_s(t; r, k) \tag{3.36}$$

Using the completeness relation (3.3a) of the reduced relative-position state and the solution of equation (3.36), we can calculate the expectation value of a physical quantity $\hat{A} = A[\hat{x}, \hat{p}]$ of the relevant system as

$$\begin{aligned}\langle A(t)\rangle &= \langle\psi(t)|A(\hat{x}, \hat{p})|\psi(t)\rangle \\ &= \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk \\ &\quad \times \psi_s^*(t; r, k) A[\tfrac{1}{2}(1+s)r + i\partial_k, \tfrac{1}{2}(1-s)k - i\partial_r]\psi_s(t; r, k) \quad (3.37)\end{aligned}$$

Therefore if the phase-space function $\psi_s(t; r, k)$ is obtained at any time, we can calculate all the physical quantities of the relevant system.

Let us now consider the case in which the system is described by a statistical operator $\hat{W}(t)$. The statistical operator $\hat{W}(t)$ can be expressed as

$$\hat{W}(t) = \hat{\rho}(t) \otimes |\phi; B\rangle\langle B; \phi| \tag{3.38}$$

where $\hat{\rho}(t)$ is a statistical operator of the relevant system at time t. The time evolution of the statistical operator is governed by the Liouville–von Neumann equation

$$\partial_t \hat{W}(t) = -i[\hat{H}, \hat{W}(t)] \tag{3.39}$$

The phase-space function that represents the statistical operator is given by

$$\begin{aligned}W_s(t; r, k|r', k') &= \langle\langle\pi_s(r, k)|\hat{W}(t)|\pi_s(r', k')\rangle\rangle \\ &= \langle\pi_s(r, k)|\hat{\rho}(t)|\pi_s(r', k')\rangle \quad (3.40)\end{aligned}$$

Thus, using equations (3.7), (3.39), and (3.40), we can obtain the equation of motion for the phase-space function $W_s(t; r, k|r', k')$ of the relevant system:

$$\begin{aligned}&\partial_t W_s(t; r, k|r', k') \\ &\quad = -i\{H[\tfrac{1}{2}(1+s)r + i\partial_k, \tfrac{1}{2}(1-s)k - i\partial_r] \\ &\qquad - H[\tfrac{1}{2}(1+s)r' - i\partial_{k'}, \tfrac{1}{2}(1-s)k' + i\partial_{r'}]\} W_s(t; r, k|r', k') \quad (3.41)\end{aligned}$$

Using the procedure given in Torres-Vega and Frederick (1993), we can rewrite this equation as

$$\begin{aligned}&\partial_t W_s(t; r, k|r', k') \\ &= -i\left\{H\left[\frac{1}{2}(1+s)r, \frac{1}{2}(1-s)k\right] \exp\left[\frac{2i}{1+s}\overleftarrow{\partial}_r \overrightarrow{\partial}_k - \frac{2i}{1-s}\overleftarrow{\partial}_k \overrightarrow{\partial}_r\right]\right. \\ &\quad \left. - H\left[\frac{1}{2}(1+s)r', \frac{1}{2}(1-s)k'\right] \exp\left[-\frac{2i}{1+s}\overleftarrow{\partial}_{r'} \overrightarrow{\partial}_{k'} + \frac{2i}{1-s}\overleftarrow{\partial}_{k'} \overrightarrow{\partial}_{r'}\right]\right\} \\ &\quad \times W_s(t; r, k|r', k') \quad (3.42)\end{aligned}$$

where we have defined the operations $\overleftarrow{\partial}_x$ and $\overrightarrow{\partial}_x$ as

$$f(x)\overleftarrow{\partial}_x = \overrightarrow{\partial}_x f(x) = \frac{\partial}{\partial x} f(x) \tag{3.43}$$

for an arbitrary differentiable function $f(x)$, and where we have further the relation

$$f(a + b\partial_x, c + d\partial_y) = f(a, c) \exp(b\overleftarrow{\partial}_a\overrightarrow{\partial}_x + d\overleftarrow{\partial}_c\overrightarrow{\partial}_y) \tag{3.44}$$

From equation (3.42) we obtain the equation of motion for the phase-space probability distribution function $W(t; r, k) = W_s(t; r, k \mid r, k)$:

$$\begin{aligned} &\partial_t W(t; r, k) \\ &\quad = 2H\left[\frac{1}{2}(1 + s)r, \frac{1}{2}(1 - s)k\right] \\ &\qquad \times \operatorname{Im}\left\{\exp\left[\frac{2i}{1 + s}\overleftarrow{\partial}_r\overrightarrow{\partial}_k - \frac{2i}{1 - s}\overleftarrow{\partial}_k\overrightarrow{\partial}_r\right] W_s(t; r, k \mid r', k')\right\}\Bigg|_{r'=r, k'=k} \end{aligned} \tag{3.45}$$

where Im stands for the imaginary part. The physical meaning of the phase-space probability distribution function will be considered in Section 4.

By making use of the phase-space function $W_s(t; r, k \mid r', k')$, we can calculate the expectation value of the physical quantity $\hat{A} = A(\hat{x}, \hat{p})$ of the relevant system as

$$\begin{aligned} \langle \hat{A}(t) \rangle &= \operatorname{Tr}[A(\hat{x}, \hat{p})\hat{W}(t)] \\ &= \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk \\ &\qquad \times A[\tfrac{1}{2}(1 + s)r + i\partial_k, \tfrac{1}{2}(1 - s)k - i\partial_r] W_s(t; r, k \mid r'k')|_{r'=r, k'=k} \end{aligned} \tag{3.46}$$

Thus, if we obtain the phase-space function $W_s(t; r, k \mid r', k')$, we can calculate all physical quantities of the relevant system.

Finally, let us briefly consider a phase-space function that represents a dynamical variable $\hat{A}$ of the relevant system. This function is given by

$$A_s(r, k \mid r', k') = \langle \pi_s(r, k) \mid \hat{A} \mid \pi_s(r', k') \rangle \tag{3.47}$$

Using the phase-space functions $A_s(r, k \mid r', k')$ and $W_s(r, k \mid r', k')$ [or $\psi_s(r, k)$], we can calculate the expectation value of $\hat{A}$ as

$$\langle \hat{A} \rangle = \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dr' \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk'$$

$$\times\ \psi_s^*(r, k) A_s(r, k \mid r', k') \psi_s(r', k') \tag{3.48}$$

for a pure state of the relevant system and as

$$\langle \hat{A} \rangle = \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dr' \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk'$$

$$\times\ A_s(r, k \mid r', k') W_s(r', k' \mid r, k) \tag{3.49}$$

for a mixed state of the relevant system.

We have used the reduced relative-position states to consider the quantum dynamics of the relevant system in the phase space. In the same way, we can use the reduced relative-momentum states to consider the quantum dynamics of the relevant system in the phase space.

3.4. Free Particle in the Phase Space

Consider, as a simple example of the general consideration, a free particle with unit mass in the phase space. Assume that the initial state of the particle at $t = 0$ is given by a Gaussian wave packet in the position space: $\psi(0; x) = \pi^{-1/4} \exp(-x^2/2)$. We also assume, for simplicity, that the state of the reference system is described by the Gaussian wave packet: $\phi(x) = \pi^{-1/4} \exp(-x^2/2)$. Since the Hamiltonian for the relevant system is given by $\hat{H} = \hat{p}^2/2$, we obtain the following wave function at time t:

$$\psi(t; x) = \frac{1}{\pi^{1/4}(1 + it)^{1/2}} \exp\left[-\frac{x^2}{2(1 + it)} \right] \tag{3.50}$$

from which the probability densities in the position and momentum spaces are given by

$$|\psi(t; x)|^2 = \frac{1}{[\pi(1 + t^2)]^{1/2}} \exp\left(-\frac{x^2}{1 + t^2} \right) \tag{3.51a}$$

$$|\tilde{\psi}(t; p)|^2 = \frac{1}{\sqrt{\pi}} \exp(-p^2) \tag{3.51b}$$

Using the kernel (3.25a) of the transformation between the phase space and position space, we obtain the phase-space function of the free particle:

$$\psi_s(t; r, k) = \frac{1}{[\pi(2 + it)]^{1/2}} \exp\left[-\frac{1}{2}\left(\frac{1}{2 + it}\right) r^2 - \frac{1}{2}\left(\frac{1 + it}{2 + it}\right) k^2 \right.$$

$$\left. - i\left(\frac{1 + it}{2 + it}\right) kr + i\,\frac{1 + s}{2}\, kr \right] \tag{3.52}$$

the squared amplitude of which is given by

$$|\psi_s(t;\, r,\, k)|^2 = \frac{1}{\pi(4 + t^2)^{1/2}} \exp\left[-\frac{2}{4 + t^2}\left(r - \frac{1}{2}\, kt\right)^2 - \frac{1}{2}\, k^2 \right] \tag{3.53}$$

Thus the phase-space probability distribution is Gaussian.

On the other hand, we obtain the equation of motion for $\psi_s(t;\, r,\, k)$ from equation (3.36),

$$\partial_t \psi_s(t;\, r,\, k) = -\tfrac{1}{2} i[\tfrac{1}{2}(1 - s)k - i\partial_r]^2 \psi_s(t;\, r,\, k) \tag{3.54}$$

and the initial condition is given by

$$\psi_s(0;\, r,\, k) = \frac{1}{\sqrt{2\pi}} \exp[-\tfrac{1}{4}r^2 - \tfrac{1}{4}k^2 - \tfrac{1}{2}ikr + \tfrac{1}{2}i(1 + s)kr] \tag{3.55}$$

Solving the equation of motion (3.54) with the initial condition (3.55), we also obtain the result given by equation (3.52).

Using the probability distribution function (3.53), we obtain the fluctuations of r and k in the phase space:

$$(\Delta r)_t^2 = \tfrac{1}{2}(t^2 + 2), \qquad (\Delta k)_t^2 = 1 \tag{3.56}$$

It should be noted that the fluctuations of the position and momentum calculated by the wave function are given by

$$(\Delta x)_t^2 = \tfrac{1}{2}(t^2 + 1), \qquad (\Delta p)_t^2 = \tfrac{1}{2} \tag{3.57}$$

Thus we find the relations

$$(\Delta r)_t^2 = (\Delta x)_t^2 + \tfrac{1}{2}, \qquad (\Delta k)_t^2 = (\Delta p)_t^2 + \tfrac{1}{2} \tag{3.58}$$

The enhancement of the fluctuations in the phase space is due to the quantum fluctuation of the reference system. This will be considered in Section 4.

3.5. Harmonic Oscillator in the Phase Space

Now let us investigate the phase-space function of a harmonic oscillator with unit mass and unit frequency. The Hamiltonian is given by

$$\hat{H} = \tfrac{1}{2}(\hat{p}^2 + \hat{x}^2) = \hat{a}^\dagger \hat{a} + \tfrac{1}{2} \tag{3.59}$$

We first consider the eigenstate of the Hamiltonian. In the Fock space representation the eigenstate is given by

$$|\psi_n\rangle = \frac{1}{n!}(\hat{a}^\dagger)^n|0\rangle \tag{3.60}$$

where $\hat{H}|\psi_n\rangle = (n + \frac{1}{2})|\psi_n\rangle$ and $|0\rangle$ is the vacuum state of the relevant system ($\hat{a}|0\rangle = 0$). Using the phase-space representation of the annihilation and creation operators given by equations (3.8a) and (3.8b), we can obtain the phase-space function of the energy eigenstate:

$$\begin{aligned}\psi_s(r, k|n) &= \frac{1}{\sqrt{n!}}\langle\pi_s(r, k)|(\hat{a}^\dagger)^n|0\rangle \\ &= \frac{1}{\sqrt{2\pi n!}}\left[\frac{1}{2}(\mu^* + s\mu) - \partial_\mu\right]^n \exp\left[-\frac{1}{2}|\mu|^2 + \frac{1}{4}s(\mu^2 - \mu^{*2})\right] \\ &= \frac{1}{\sqrt{2\pi n!}}\mu^{*n}\exp\left[-\frac{1}{2}|\mu|^2 + \frac{1}{4}s(\mu^2 - \mu^{*2})\right]\end{aligned} \tag{3.61}$$

with $\mu = (r + ik)/\sqrt{2}$. In deriving this equation, we have assumed, for simplicity, that the reference system is in the vacuum state. The phase-space function $\psi_s(r, k|n)$ satisfies the relations

$$\int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\, \psi_s^*(r, k|m)\psi_s(r, k|n) = \delta_{m,n} \tag{3.62a}$$

$$\sum_{n=0}^{\infty} \psi_s^*(r, k|n)\psi_s(r', k'|n) = \frac{1}{2\pi}e^{-is(kr-k'r')/2}\langle\mu'|\mu\rangle \tag{3.62b}$$

where $|\mu\rangle$ is a coherent state of the relevant system.

The phase-space probability distribution function is given by the squared amplitude of the phase-space function. Thus we obtain from equation (3.61)

$$|\psi_s(r, k|n)|^2 = \frac{1}{2\pi n!}\left[\frac{1}{2}(r^2 + k^2)\right]^n \exp\left[-\frac{1}{2}(r^2 + k^2)\right] \tag{3.63}$$

It should be noted here that since the classical Hamiltonian of the harmonic oscillator is given by $H_{\text{cl}}(x, p) = \frac{1}{2}(p^2 + x^2)$, we can express the probability distribution (3.63) as

$$|\psi_s(r, k|n)|^2 = \frac{1}{2\pi n!}[H_{\text{cl}}(r, k)]^n e^{-H_{\text{cl}}(r,k)} \tag{3.64}$$

From equation (3.63) or (3.64) we find that in the phase space the most probable trajectory of the energy eigenstate is characterized by

$$H_{\text{cl}}(r, k) = \frac{1}{2}(r^2 + k^2) = n \tag{3.65}$$

This relation corresponds to the semiclassical quantization condition of a harmonic oscillator.

Now consider the coherent state $|\alpha\rangle$ of the relevant system with complex amplitude $\alpha = (q + ip)/\sqrt{2}$. In this case we obtain the phase-space function,

$$\psi_s(r, k|q, p) = \langle \pi_s(r, k)|\alpha\rangle$$
$$= \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{4}(r - q)^2 - \frac{1}{4}(k - p)^2 + \frac{1}{2} i(pr - qk + skr)\right] \tag{3.66}$$

where we have assumed the vacuum state of the reference system. Since we have $\exp(-i\hat{H}t|\alpha\rangle = |\alpha \exp(-it)\rangle$ for the harmonic oscillator Hamiltonian (3.59), the phase-space function at time t is given by

$$\psi_s(t; r, k|q, p)$$
$$= \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{4}(r - q_t)^2 - \frac{1}{4}(k - p_t)^2 + \frac{1}{2} i(p_t r - q_t k + skr)\right] \tag{3.67}$$

where q_t and q_t are the solutions of the classical equations of motion for the harmonic oscillator:

$$q_t = q \cos t + p \sin t \tag{3.68a}$$

$$p_t = p \cos t - q \sin t \tag{3.68b}$$

Since the phase-space probability distribution function is obtained from equation (3.67),

$$|\psi_s(t; r, k|q, p)|^2 = \frac{1}{2\pi} \exp\left[-\frac{1}{2}(r - q_t)^2 - \frac{1}{2}(k - p_t)^2\right] \tag{3.69}$$

we find that in the phase space the most probable path of the coherent state is the classical trajectory of the harmonic oscillator.

In the same way we obtain the phase-space function for the squeezed state of the relevant system:

$$\psi_s(r, k|\gamma, q, p)$$
$$= \langle \pi_s(r, k)|\hat{D}(\alpha)\hat{S}(\gamma)|0\rangle$$
$$= \frac{1}{[\pi(e^{\gamma} + e^{-\gamma})]^{1/2}} \exp\left[-\frac{(r - q)^2}{2(e^{2\gamma} + 1)} - \frac{(k - p)^2}{2(1 + e^{-2\gamma})}\right.$$
$$\left. + i\left(\frac{pr}{1 + e^{-2\gamma}} - \frac{kq}{e^{2\gamma} + 1}\right) - \frac{1}{2} i \frac{e^{2\gamma} - 1}{e^{2\gamma} + 1}(kr + pq) + \frac{1}{2} iskr\right] \tag{3.70}$$

where the displacement operator $\hat{D}(\alpha)$ and the squeezing operator $\hat{S}(\gamma)$ are given by

$$\hat{D}(\alpha) = \exp[\alpha\hat{a}^\dagger - \alpha^*\hat{a}], \qquad \hat{S}(\gamma) = \exp[\tfrac{1}{2}\gamma(\hat{a}^{\dagger 2} - \hat{a}^2)] \tag{3.71}$$

and we have assumed that the squeezing parameter γ is real. The phase-space probability distribution of the squeezed state at time t is given by

$$\begin{aligned} &|\psi_s(t; r, k|\gamma, q, p)|^2 \\ &\quad = \frac{1}{\pi(e^\gamma + e^{-\gamma})}\exp\left[-\frac{1 + (e^{2\gamma} - 1)\sin^2 t}{e^{2\gamma} + 1}(r - q_t)^2\right. \\ &\quad\quad - \frac{1 - (1 - e^{-2\gamma})\sin^2 t}{1 + e^{-2\gamma}}(k - p_t)^2 \\ &\quad\quad \left. - \frac{(e^{2\gamma} - 1)\sin 2t}{e^{2\gamma} + 1}(r - q_t)(k - p_t)\right] \end{aligned} \tag{3.72}$$

where q_t and p_t are given by equations (3.68a) and (3.68b). The average values and fluctuations of r and k at time t are calculated as

$$\langle r\rangle_t = q_t, \qquad \langle k\rangle_t = p_t \tag{3.73a}$$

$$(\Delta r)_t^2 = \tfrac{1}{2}(1 + e^{2\gamma})[\cos^2 t + e^{-2\gamma}\sin^2 t] \tag{3.73b}$$

$$(\Delta k)_t^2 = \tfrac{1}{2}(1 + e^{-2\gamma})[\cos^2 t + e^{2\gamma}\sin^2 t] \tag{3.73c}$$

We also find that the most probable path of the squeezed state in the phase space is the classical trajectory.

Finally, consider the thermal state of the harmonic oscillator as an example of the mixed state. The statistical operator $\hat{\rho}$ of the thermal state is given by

$$\hat{\rho} = \frac{1}{1 + \bar{n}}\sum_{n=0}^{\infty}|n\rangle\left(\frac{\bar{n}}{1 + \bar{n}}\right)^n\langle n| \tag{3.74}$$

where $|n\rangle$ is the number eigenstate and $\bar{n}$ is a bosonic distribution function. We assume here that the reference system is in the vacuum state. Thus the phase-space function $W(r, k|r', k')$ is calculated as

$$\begin{aligned} &W_s(r, k|r', k') \\ &= \frac{1}{1 + \bar{n}}\sum_{n=0}^{\infty}\left(\frac{\bar{n}}{1 + \bar{n}}\right)^n \psi_s(r, k|n)\psi_s^*(r', k'|n) \\ &= \frac{1}{2\pi(1 + \bar{n})}\exp\left[-\frac{rr' + kk'}{2(1 + \bar{n})} - \frac{i(rk' - r'k)}{2(1 + \bar{n})} - \frac{1}{4}(r - r')^2 - \frac{1}{4}(k - k')^2\right. \\ &\quad \left. - \frac{i}{2}(k - k')(r + r') + \frac{i}{2}(s + 1)(kr - k'r')\right] \end{aligned} \tag{3.75}$$

where we have used the phase-space function (3.61). Using this result, we obtain the phase-space probability distribution function $W(r, k) = W(r, k \mid r, k)$ for the thermal state:

$$W(r, k) = \frac{1}{2\pi(1 + \bar{n})} \exp\left[-\frac{r^2 + k^2}{2(1 + \bar{n})} \right]$$
$$= \frac{1}{2\pi(1 + \bar{n})} \exp\left[-\frac{H_{\text{cl}}(r, k)}{1 + \bar{n}} \right] \tag{3.76}$$

where $H_{\text{cl}}(r, k)$ is the classical Hamiltonian of the harmonic oscillator. To check the formula (3.46), let us calculate the thermal average of the Hamiltonian (3.59):

$$\langle \hat{H} \rangle = \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk \, \frac{1}{2} \left\{ \left[\frac{1}{2} (1 + s)r + i\partial_k \right]^2 + \left[\frac{1}{2} (1 - s)k - i\partial_r \right]^2 \right\}$$
$$\times W_s(r, k \mid r', k') \mid_{r'=r, k'=k}$$
$$= \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk \left[\frac{1}{2} + \frac{\bar{n}}{1 + \bar{n}} H_{\text{cl}}(r, k) \right] W(r, k)$$
$$= \frac{1}{2} + \bar{n} \tag{3.77}$$

where $W_s(r, k \mid r', k')$ and $W(r, k)$ are given by equations (3.75) and (3.76). This is the expected result. It is easy to see from equations (3.76) and (3.77) that the following relation is satisfied:

$$\langle \hat{H} \rangle = \langle H_{\text{cl}}(r, k) \rangle + \tfrac{1}{2} \tag{3.78}$$

The second term on the right-hand side of this equation is the vacuum energy.

4. PROBABILITY DISTRIBUTION IN THE PHASE SPACE

4.1. Phase-Space Probability Distributions

In this section, phase-space probability distributions are introduced in terms of the relative-position states and relative-momentum states, and their physical meanings are considered in detail. The phase-space probability distribution considered in this section is shown to be closely related to the operational probability distribution (Wódkiewicz, 1984, 1986, 1987; Royer, 1985; Burak and Wódkiewicz, 1992). We assume here that the state of the composite system (the relevant and reference systems) is described by the noncorrelated statistical operator

$$\hat{W} = \hat{\rho} \otimes \hat{\rho}' \tag{4.1}$$

where $\hat{\rho}$ and $\hat{\rho}'$ are, respectively, the statistical operators of the relevant and reference systems. The phase-space probability distributions $W(r, k)$ and $\tilde{W}(r, k)$ in the relative-state formulation are defined as

$$W(r, k) = \langle\langle \pi_s(r, k) | \hat{W} | \pi_s(r, k) \rangle\rangle \tag{4.2a}$$

$$\tilde{W}(r, k) = \langle\langle \tilde{\pi}_s(r, k) | \hat{W} | \tilde{\pi}_s(r, k) \rangle\rangle \tag{4.2b}$$

where the relative-position $|\pi_s(r, k)\rangle\rangle$ and relative-momentum states $|\tilde{\pi}_s(r, k)\rangle\rangle$ of the composite system are given by equations (2.8) and (2.19), respectively. It should be noted that the phase-space probability distributions are positive-definite, independent of the parameter s, and normalized as

$$\int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\ W(r, k) = \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\ \tilde{W}(r, k) = 1 \tag{4.3}$$

Since the relative-position state $|\pi_s(r, k)\rangle\rangle$ is the simultaneous eigenstate of operators $\hat{x}_A - \hat{x}_B$ and $\hat{p}_A + \hat{p}_B$ and the relative-momentum state $|\tilde{\pi}_s(r, k)\rangle\rangle$ is the simultaneous eigenstate of operators $\hat{x}_A + \hat{x}_B$ and $\hat{p}_A - \hat{p}_B$, we obtain the relations

$$\langle (\hat{x}_A - \hat{x}_B)^m (\hat{p}_A + \hat{p}_B)^n \rangle = \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\ r^m k^n W(r, k) \tag{4.4a}$$

$$\langle (\hat{x}_A + \hat{x}_B)^m (\hat{p}_A - \hat{p}_B)^n \rangle = \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\ r^m k^n \tilde{W}(r, k) \tag{4.4b}$$

where we set $\langle \hat{O} \rangle = \mathrm{Tr}_A\, \mathrm{Tr}_B[\hat{O}\hat{\rho} \otimes \hat{\rho}']$ for any operator $\hat{O}$.

First consider the phase-space probability distribution $W(r, k)$ in terms of the relative-position states. Using the definition of the relative-position state $|\pi_s(r, k)\rangle\rangle$, we find the expression

$$W(r, k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\ e^{ik(x-y)} \langle A; y + r | \hat{\rho} | x + r; A \rangle \langle B; y | \hat{\rho}' | x; B \rangle \tag{4.5}$$

Now, introduce an operator $\hat{\rho}_{\mathrm{ref}}$ of the relevant system through the relation

$$\langle A; x | \hat{\rho}_{\mathrm{ref}} | y; A \rangle = \langle B; y | \hat{\rho}' | x; B \rangle \tag{4.6}$$

where the operator $\hat{\rho}_{\mathrm{ref}}$ satisfies the property of a statistical operator. It should be noted here that the quantity on the left-hand side of this equation is calculated in the relevant system and the quantity on the right-hand side is calculated in the reference system. It is easy to see that the operator $\hat{\rho}_{\rm ref}$ can be expressed as

$$\hat{\rho}_{\rm ref} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |m; A\rangle[\langle B; n|\hat{\rho}'|m; B\rangle]\langle A; n| \tag{4.7}$$

Then using the operator $\hat{\rho}_{\rm ref}$, we can rewrite equation (4.5) as

$$\begin{aligned} W(r, k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \\ &\quad \times e^{ik(x-y)}\langle A; y + r|\hat{\rho}|x + r; A\rangle\langle A; x|\hat{\rho}_{\rm ref}|y; A\rangle \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \\ &\quad \times \langle A; y|e^{ir\hat{p}_A}\hat{\rho}e^{-ir\hat{p}_A}|x; A\rangle\langle A; x|e^{ik\hat{x}_A}\hat{\rho}_{\rm ref}e^{-ik\hat{x}_A}|y; A\rangle \\ &= \frac{1}{2\pi} \mathrm{Tr}[e^{ir\hat{p}_A}\hat{\rho}e^{-ir\hat{p}_A}e^{ik\hat{x}_A}\hat{\rho}_{\rm ref}e^{-ik\hat{x}_A}] \\ &= \frac{1}{2\pi} \mathrm{Tr}[\hat{\rho}\hat{D}(r, k)\hat{\rho}_{\rm ref}\hat{D}^\dagger(r, k)] \end{aligned} \tag{4.8}$$

where $\hat{x}_A$ and $\hat{p}_A$ are the position and momentum operators of the relevant system and Tr stands for taking the trace of the relevant system. In this equation, $\hat{D}(r, k)$ is the displacement operator in the phase space:

$$\begin{aligned} \hat{D}(r, k) &= \exp[i(k\hat{x}_A - r\hat{p}_A)] \\ &= \exp[\mu\hat{a}^\dagger - \mu^*\hat{a}] = \hat{D}(\mu) \end{aligned} \tag{4.9}$$

where the complex parameter μ is given by $\mu = (r + ik)/\sqrt{2}$ and where $\hat{a} = (\hat{x}_A + i\hat{p}_A)/\sqrt{2}$ is the annihilation operator of the relevant system. The phase-space probability distributions for several states of the reference system are given in Section 4.6.

Suppose that both the relevant and reference systems are in the pure states and that we set $\hat{\rho} = |\psi; A\rangle\langle A; \psi|$ and $\hat{\rho}' = |\phi^*; B\rangle\langle B; \phi^*|$. In this case the relation (4.6) becomes

$$\langle A; x|\hat{\rho}_{\rm ref}|y; A\rangle = \langle B; y|\phi^*; B\rangle\langle B; \phi^*|x; B\rangle \tag{4.10}$$

and so the operator $\hat{\rho}_{\rm ref}$ is given by $\hat{\rho}_{\rm ref} = |\phi; A\rangle\langle A; \phi|$. Substituting $\hat{\rho}$ and $\hat{\rho}_{\rm ref}$ into equation (4.8), we obtain the probability distribution

$$W(r, k) = \frac{1}{2\pi} |\langle\psi|\hat{D}(r, k)|\phi\rangle|^2 \tag{4.11}$$

where we have dropped the index A. This probability distribution is equivalent to that obtained by Wódkiewicz (1984, 1986), who considered the operational meanings of the measurement process. Furthermore, the probability distribution (4.11) can be written as

$$W(r, k) = \frac{1}{2\pi}\left|\int_{-\infty}^{\infty} dx\, \psi(x + r)\phi(x)e^{-ikx}\right|^2 \tag{4.12}$$

where $\psi(x) = \langle x|\psi\rangle$ and $\phi(x) = \langle x|\phi\rangle$. The physical meaning of the quantity on the right-hand side was discussed first by Aharonov *et al.* (1982; O'Connell and Rajagopal, 1982). We can thus see that the phase-space probability distribution $W(r, k)$ constructed in terms of the relative-position states is closely related to the operational probability distribution.

Next we introduce, after Cahill and Glauber (1969a,b), the following quantities of the relevant system:

$$W(\alpha; \epsilon) = \mathrm{Tr}[\hat{\rho}\hat{T}(\alpha; \epsilon)] \tag{4.13a}$$

$$\hat{T}(\alpha; \epsilon) = \frac{1}{\pi}\int_{\mathbf{R}^2} d^2\xi\, \hat{D}(\xi; \epsilon)e^{\alpha\xi^* - \alpha^*\xi} \tag{4.13b}$$

$$\hat{D}(\xi; s) = \exp(\xi\hat{a}^\dagger - \xi^*\hat{a} + \tfrac{1}{2}\epsilon|\xi|^2) \tag{4.13c}$$

where $d^2\xi = d(\mathrm{Re}\,\xi)d(\mathrm{Im}\,\xi)$. The function $W(\alpha; \epsilon)$ is the ϵ-ordered quasiprobability distribution. The statistical operator can be expressed in terms of these quantities as

$$\hat{\rho} = \frac{1}{\pi}\int_{\mathbf{R}^2} d^2\alpha\, W(\alpha; -\epsilon)\hat{T}(\alpha; \epsilon) \tag{4.14}$$

Similarly, the operator $\hat{\rho}_{\mathrm{ref}}$ corresponding to the state of the reference system can be expressed as

$$\hat{\rho}_{\mathrm{ref}} = \frac{1}{\pi}\int_{\mathbf{R}^2} d^2\alpha\, W_{\mathrm{ref}}(\alpha; -\epsilon)\hat{T}(\alpha; \epsilon) \tag{4.15}$$

where $W_{\mathrm{ref}}(\alpha; \epsilon)$ is given by replacing $\hat{\rho}$ with $\hat{\rho}_{\mathrm{ref}}$ in equation (4.13a).

Substituting equations (4.14) and (4.15) into equation (4.8) and using the relation

$$\hat{D}(\mu)\hat{T}(\alpha; \epsilon)\hat{D}^\dagger(\mu) = \hat{T}(\mu + \alpha; \epsilon) \tag{4.16}$$

we can obtain the phase-space probability distribution in terms of the ϵ-ordered quasiprobability distributions $W(\alpha, \epsilon)$ and $W_{\mathrm{ref}}(\alpha, \epsilon)$:

$$W(r, k) = \frac{1}{2\pi^2}\int_{\mathbf{R}^2} d^2\alpha\, W_{\mathrm{ref}}(\alpha; -\epsilon)W(\alpha + \mu; \epsilon) \tag{4.17}$$

with $\mu = (r + ik)/\sqrt{2}$. It should be noted that if we set $\epsilon = 1, 0$, or -1, the function $W(\alpha; \epsilon)$ becomes the P-function, Wigner function, or Q-function:

$$P(\alpha) = \frac{1}{\pi} W(\alpha; 1) = \frac{1}{\pi^2} \int_{\mathbf{R}^2} d^2\xi \, \mathrm{Tr}[\hat{\rho} e^{\xi \hat{a}^\dagger} e^{-\xi^* \hat{a}}] e^{\alpha \xi^* - \alpha^* \xi} \tag{4.18}$$

$$W_W(\alpha) = \frac{1}{\pi} W(\alpha; 0) = \frac{1}{\pi^2} \int_{\mathbf{R}^2} d^2\xi \, \mathrm{Tr}[\hat{\rho} e^{\xi \hat{a}^\dagger - \xi^* \hat{a}}] e^{\alpha \xi^* - \alpha^* \xi} \tag{4.19}$$

$$Q(\alpha) = \frac{1}{\pi} W(\alpha; -1) = \frac{1}{\pi^2} \int_{\mathbf{R}^2} d^2\xi \, \mathrm{Tr}[\hat{\rho} e^{-\xi^* \hat{a}} e^{\xi \hat{a}^\dagger}] e^{\alpha \xi^* - \alpha^* \xi} \tag{4.20}$$

Thus we can express the phase-space probability distribution (4.17) as

$$W(r, k) = \frac{1}{2} \int_{\mathbf{R}^2} d^2\alpha \, P_{\mathrm{ref}}(\alpha) Q(\alpha + \mu) \tag{4.21a}$$

$$= \frac{1}{2} \int_{\mathbf{R}^2} d^2\alpha \, Q_{\mathrm{ref}}(\alpha) P(\alpha + \mu) \tag{4.21b}$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp \, W_{\mathrm{W,ref}}(x, p) W_W(x + r, p + k) \tag{4.21c}$$

where $P(\alpha)$ $[P_{\mathrm{ref}}(\alpha)]$, $Q(\alpha)$ $[Q_{\mathrm{ref}}(\alpha)]$, and $W_W(x, p)$ $[W_{\mathrm{W,ref}}(x, p)]$ are the P-function, Q-function, and Wigner function, respectively, corresponding to $\hat{\rho}$ $[\hat{\rho}_{\mathrm{ref}}]$. Thus, we can use these relations to obtain the phase-space probability distribution at any time if we know the time evolution for the P-function, Q-function, or Wigner function. When we know, for example, the Wigner functions of the relevant and reference systems at time t, $W_W(t; x, p)$ and $W_{\mathrm{W,ref}}(t; x, p)$, we can use the relation (4.21c) to obtain the phase-space probability distribution $W(t; r, k)$ at time t. It should be noted here that the normalization conditions are given by

$$\int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk \, W(r, k) = 2 \int_{\mathbf{R}^2} d^2\mu \, W(r, k) = 1 \tag{4.22a}$$

$$\int_{\mathbf{R}^2} d^2\alpha \, P(\alpha) = \int_{\mathbf{R}^2} d^2\alpha \, Q(\alpha) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp \, W_W(x, p) = 1 \tag{4.22b}$$

The right-hand side of equation (4.21c) has the same form as the nonnegative smoothed Wigner function introduced in Lalović *et al.* (1992).

Now consider the phase-space probability distribution $\tilde{W}(r, k)$ constructed in terms of the relative-momentum states. Let us introduce an operator $\hat{\sigma}_{\mathrm{ref}}$ of the relevant system by the relation

$$\langle A; p|\hat{\sigma}_{\text{ref}}|p'; A\rangle = \langle B; p'|\hat{\rho}'|p; B\rangle \tag{4.23}$$

where $|p; A\rangle$ and $|p; B\rangle$ are the momentum eigenstates of the relevant and reference systems, and where the operator $\hat{\sigma}_{\text{ref}}$ has the property of a statistical operator. The operator $\hat{\sigma}_{\text{ref}}$ can be expressed as

$$\hat{\sigma}_{\text{ref}} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} |m; A\rangle[(-1)^{m+n}\langle B; n|\hat{\rho}'|m; B\rangle]\langle A; n| \tag{4.24}$$

Using the operator $\hat{\sigma}_{\text{ref}}$, we can express the phase-space probability distribution $\tilde{W}(r, k)$ as

$$\tilde{W}(r, k) = \frac{1}{2\pi}\,\text{Tr}[\hat{\rho}\hat{D}(r, k)\hat{\sigma}_{\text{ref}}\hat{D}^{\dagger}(r, k)] \tag{4.25}$$

where $\hat{D}(r, k)$ is the displacement operator given by equation (4.9). We can therefore obtain results for $\tilde{W}(r, k)$ by replacing $\hat{\rho}_{\text{ref}}$ with $\hat{\sigma}_{\text{ref}}$ in those obtained for $W(r, k)$.

Finally, consider the relation between the operators $\hat{\rho}_{\text{ref}}$ and $\hat{\sigma}_{\text{ref}}$ given by equations (4.6) and (4.23). Using the fact that $|x; A\rangle$ and $|p; A\rangle$ are connected to each other by the Fourier transformation, we can obtain the relations

$$\langle A; x|\hat{\sigma}_{\text{ref}}|x'; A\rangle = \langle A; -x|\hat{\rho}_{\text{ref}}|-x'; A\rangle \tag{4.26a}$$

$$\langle A; p|\hat{\sigma}_{\text{ref}}|p'; A\rangle = \langle A; -p|\hat{\rho}_{\text{ref}}|-p'; A\rangle \tag{4.26b}$$

Thus, we can use the parity operator $\hat{\Pi}$ introduced in Royer (1977) and obtain the relations

$$\hat{\sigma}_{\text{ref}} = \hat{\Pi}\hat{\rho}_{\text{ref}}\hat{\Pi}, \qquad \hat{\rho}_{\text{ref}} = \hat{\Pi}\hat{\sigma}_{\text{ref}}\hat{\Pi} \tag{4.27}$$

Here the parity operator $\hat{\Pi}$ of the relevant system is given by

$$\hat{\Pi} = \int_{-\infty}^{\infty} dx\, |-x; A\rangle\langle A; x| = \int_{-\infty}^{\infty} dp\, |-p; A\rangle\langle A; p| \tag{4.28}$$

which satisfies the relations $\hat{\Pi} = \hat{\Pi}^{\dagger}$ and $\hat{\Pi}^2 = \hat{1}$. In particular, we find that $\hat{\rho}_{\text{ref}} = \hat{\sigma}_{\text{ref}}$ for the vacuum state of the reference system.

4.2. Marginal Distributions

Now consider the marginal distributions derived from the phase-space probability distributions $W(r, k)$ and $\tilde{W}(r, k)$. The marginal distributions of the quasiprobabilities in quantum optics have recently been investigated in order to get a deeper understanding of the nonclassical photon state (Agarwal, 1993; Orlowski and Wünsche, 1993). For the phase-space distribution function $W(r, k)$ defined by equation (4.2a), the marginal distributions are given by

$$W(r) = \int_{-\infty}^{\infty} dk\, W(r, k) \tag{4.29a}$$

$$W(k) = \int_{-\infty}^{\infty} dr\, W(r, k) \tag{4.29b}$$

Substituting equation (4.5) into these equations, we can express the marginal distributions $W(r)$ and $W(k)$ as

$$W(r) = \int_{-\infty}^{\infty} dx\, f(x - r)\langle A; x|\hat{\rho}|x; A\rangle \tag{4.30a}$$

$$W(k) = \int_{-\infty}^{\infty} dp\, g(k - p)\langle A; p|\hat{\rho}|p; A\rangle \tag{4.30b}$$

where the functions $f(x)$ and $g(p)$ are given by

$$f(x) = \langle A; x|\hat{\rho}_{\text{ref}}|x; A\rangle = \langle B; x|\hat{\rho}'|x; B\rangle \tag{4.31a}$$

$$g(p) = \langle A; p|\hat{\sigma}_{\text{ref}}|p; A\rangle = \langle B; p|\hat{\rho}'|p; B\rangle \tag{4.31b}$$

Here we have used the relations (4.6) and (4.23). We see from equations (4.30a) and (4.31a) or equations (4.30b) and (4.31b) that in the position or momentum measurement, the function $f(x)$ or $g(p)$ plays the role of the filter function that determines the accuracy of the measured value. The filter function is determined if the state of the reference system is given. Thus it seems reasonable to consider that the reference system is the measurement apparatus. Furthermore, it should be noted that the marginal distributions (4.30a) and (4.30b) can be derived by means of the fuzzy space formulation of quantum mechanics (Prugovečki, 1976a,b, 1978; Twareque Ali and Prugovečki, 1977).

When the reference system is in the coherent state $|\alpha\rangle$ with complex amplitude $\alpha = (q + ip)/\sqrt{2}$, we obtain the filter functions

$$f(x) = \pi^{-1/2} \exp[-(x - q)^2] \tag{4.32a}$$

$$g(k) = \pi^{-1/2} \exp[-(k - p)^2] \tag{4.32b}$$

This means that the coherent state is the optimal state for simultaneous measurement of position and momentum. For the squeezed-vacuum state of the reference system $|\gamma\rangle = \exp[\frac{1}{2}\gamma(\hat{b}^{\dagger 2} - \hat{b}^2)]|0\rangle$ with real squeezing parameter γ, the filter functions $f(x)$ and $g(k)$ are given by

$$f(x) = (e^{2\gamma}\pi)^{-1/2} \exp(-e^{-2\gamma}x^2) \tag{4.33a}$$

$$g(k) = (e^{-2\gamma}\pi)^{-1/2} \exp(-e^{2\gamma}k^2) \tag{4.33b}$$

Thus if $\gamma > 0$, the measurement of momentum is more accurate than that of position. In particular, when the squeezing parameter is extremely large, we can approximate $W(k) \approx \langle A; k|\hat{\rho}|k; A\rangle$ for $\gamma >> 1$ and $W(r) \approx \langle A; r|\hat{\rho}|r; A\rangle$ for $-\gamma >> 1$. This indicates that one of the marginal distributions approaches the position or momentum probability of the relevant system if we use the highly squeezed vacuum state of the reference system.

Next consider the phase-space distribution function $\tilde{W}(r, k)$ defined by equation (4.2b). The marginal distributions are given by

$$\tilde{W}(r) = \int_{-\infty}^{\infty} dk\, \tilde{W}(r, k) = \int_{-\infty}^{\infty} dx\, f(r - x)\langle A; x|\hat{\rho}|x; A\rangle \tag{4.34a}$$

$$\tilde{W}(k) = \int_{-\infty}^{\infty} dr\, \tilde{W}(r, k) = \int_{-\infty}^{\infty} dp\, g(p - k)\langle A; p|\hat{\rho}|p; A\rangle \tag{4.34b}$$

where the functions $f(x)$ and $g(p)$ are given by equations (4.31a) and (4.31b). The meanings of these marginal distributions are the same as those of the distributions (4.30a) and (4.30b). If $f(x)$ [or $g(p)$] is an even function, the marginal distributions obtained from the phase-space functions $W(r, k)$ and $\tilde{W}(r, k)$ are equal to each other. That is, $W(r) = \tilde{W}(r)$ [or $W(k) = \tilde{W}(k)$].

4.3. Characteristic Function

Here we obtain the characteristic function by means of the phase-space probability distribution. We will confine ourselves to considering the function $W(r, k)$, but the characteristic function derived from the function $\tilde{W}(r, k)$ can be obtained in the same way. Using equations (4.13) and (4.17), we can obtain the relation

$$\int_{\mathbf{R}^2} d^2\mu\, W(\mu)e^{-\mu\alpha^* + \mu^*\alpha} = \frac{1}{2}\, C(\alpha; \epsilon)C_{\text{ref}}(-\alpha; -\epsilon) \tag{4.35}$$

where the characteristic functions $C(\alpha; \epsilon)$ and $C_{\text{ref}}(\alpha; \epsilon)$ are given by

$$C(\alpha; \epsilon) = \text{Tr}[\hat{\rho}\hat{D}(\alpha; \epsilon)] = \frac{1}{\pi}\int_{\mathbf{R}^2} d^2\beta\, W(\beta; \epsilon)e^{-\alpha\beta^* + \alpha^*\beta} \tag{4.36a}$$

$$C_{\text{ref}}(\alpha; \epsilon) = \text{Tr}[\hat{\rho}_{\text{ref}}\hat{D}(\alpha; \epsilon)] = \frac{1}{\pi}\int_{\mathbf{R}^2} d^2\beta\, W_{\text{ref}}(\beta; \epsilon)e^{-\alpha\beta^* + \alpha^*\beta} \tag{4.36b}$$

Then using equation (4.35), we can calculate the moment $\langle r^m k^n \rangle$ as

$$\begin{aligned}
\langle r^m k^n \rangle &= \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\, r^m k^n W(r, k) \\
&= 2^{1+(m+n)/2} \int_{\mathbf{R}^2} d^2\mu\, \mu_1^m \mu_2^n W(\mu) \\
&= i^{n-m} 2^{1-(m+n)/2} \int_{\mathbf{R}^2} d^2\mu\, W(\mu) \partial_{\alpha_2}^m \partial_{\alpha_1}^n e^{-\mu\alpha^* + \mu^*\alpha} |_{\alpha=0} \\
&= i^{n-m} 2^{-(m+n)/2} [\partial_{\alpha_2}^m \partial_{\alpha_1}^n C(\alpha; \epsilon) C_{\text{ref}}(-\alpha; -\epsilon)] |_{\alpha=0} \qquad (4.37)
\end{aligned}$$

where $\mu = \mu_1 + i\mu_2$, $\alpha = \alpha_1 + i\alpha_2$, and $\partial_{\alpha_j} = \partial/\partial\alpha_j$, and where we have used the fact that $\mu = (r + ik)/\sqrt{2}$. Therefore if the respective characteristic functions of the relevant and reference systems are given, we can use equation (4.37) to obtain all the moments in the phase space. Thus we obtain the characteristic function in the phase space:

$$\langle e^{i(\xi r - \zeta k)} \rangle = C\left(\frac{\zeta + i\xi}{\sqrt{2}}; \epsilon\right) C_{\text{ref}}\left(-\frac{\zeta + i\xi}{\sqrt{2}}; -\epsilon\right) \qquad (4.38)$$

That is, the characteristic function by means of the phase-space probability distribution is given as the product of the characteristic functions calculated by the quasiprobability distributions of the relevant and reference systems.

We can use equation (4.35) to express the phase-space probability distribution $W(r, k)$ as

$$W(r, k) = \frac{1}{2\pi^2} \int_{\mathbf{R}^2} d^2\alpha\, C(\alpha; \epsilon) C_{\text{ref}}(-\alpha; -\epsilon) e^{\mu\alpha^* - \mu^*\alpha} \qquad (4.39)$$

Thus it is easy to see that the phase-space probability distribution $W(r, k)$ is given by

$$W(r, k) = \frac{1}{2\pi} \tilde{C}_{\text{ref}}(\partial_{\mu^*}, -\partial_\mu; -\epsilon) W(\mu; \epsilon) \qquad (4.40)$$

where we set $\tilde{C}_{\text{ref}}(\alpha, \alpha^*; \epsilon) = C_{\text{ref}}(\alpha; \epsilon) = \text{Tr}[\hat{\rho}_{\text{ref}} \hat{D}(\alpha; \epsilon)]$. When the reference system is in the vacuum state, we obtain $C_{\text{ref}}(\alpha, \alpha^*; -\epsilon) = \exp[-(1 + \epsilon)|\alpha|^2/2]$. Equation (4.40) thus becomes

$$\begin{aligned}
W(r, k) &= \frac{1}{2\pi} \exp\left[\frac{1}{2}(1 + \epsilon)\partial_\mu \partial_{\mu^*}\right] W(\mu; \epsilon) \\
&= \frac{1}{2\pi} \exp\left[\frac{1}{4}(1 + \epsilon)(\partial_r^2 + \partial_k^2)\right] W(\mu; \epsilon) \qquad (4.41)
\end{aligned}$$

In particular, using equation (4.8) with $\hat{\rho}_{\rm ref} = |0\rangle\langle 0|$, we obtain the following expressions:

$$
\begin{aligned}
W(r, k) &= \tfrac{1}{2}Q(\mu) \\
&= \tfrac{1}{2}\exp(\partial_\mu \partial_{\mu^*})\, P(\mu) \\
&= \exp[\tfrac{1}{4}(\partial_r^2 + \partial_k^2)]W_{\rm W}(r, k)
\end{aligned}
\tag{4.42}
$$

where $Q(\mu)$, $P(\mu)$, and $W_{\rm W}(r, k)$ are the Q-function, P-function, and Wigner function, respectively, of the relevant system.

4.4. Uncertainty Relation

This section briefly considers the uncertainty relation in the phase space. It is easy to see that the moments $\langle r^n\rangle$ and $\langle k^n\rangle$ are calculated as

$$\langle r^n\rangle = \int_{-\infty}^{\infty} dr\, r^n W(r) = \mathrm{Tr}[(\hat{x}_A - \hat{x}_B)^n \hat{\rho} \otimes \hat{\rho}'] \tag{4.43a}$$

$$\langle k^n\rangle = \int_{-\infty}^{\infty} dk\, k^n W(k) = \mathrm{Tr}[(\hat{p}_A + \hat{p}_B)^n \hat{\rho} \otimes \hat{\rho}'] \tag{4.43b}$$

where $W(r)$ and $W(k)$ are the marginal distributions given by equations (4.29a) and (4.29b). Using equations (4.43a) and (4.43b), we can obtain the uncertainty relation

$$(\Delta r)^2(\Delta k)^2 \geq [(\Delta x_A)(\Delta p_A) + (\Delta x_B)(\Delta p_B)]^2 \geq 1 \tag{4.44}$$

where we set $(\Delta A)^2 = \langle A^2\rangle - \langle A\rangle^2$. In deriving the inequality, we used the fact that there is no correlation between the relevant and reference systems. The minimum value of the uncertainty product is twice that of the usual uncertainty product. This relation is equivalent to the operational uncertainty relation obtained in Wódkiewicz (1987).

Although the uncertainty product $\Delta r \Delta k$ has to satisfy the inequality (4.44), we can reduce the fluctuation such that $\Delta r < 1$ or $\Delta k < 1$. This may be called operational squeezing. In fact, when the relevant system is in the coherent state and the reference system is in the squeezed-vacuum state, we can obtain $(\Delta r)^2 = \frac{1}{2}(1 + e^{2\gamma})$ and $(\Delta k)^2 = \frac{1}{2}(1 + e^{-2\gamma})$ (see Section 4.6). If the squeezing parameter of the reference system is extremely large, the additional noise due to the quantum fluctuation of the reference system can be taken out completely from that of the phase-space variables. If both relevant and reference systems are in the coherent states, we obtain the relation $\Delta r = \Delta k = 1$.

It should be noted that $\hat{x}_A - \hat{x}_B$ commutes with $\hat{p}_A + \hat{p}_B$. Thus, $\hat{x}_A - \hat{x}_B$ and $\hat{p}_A + \hat{p}_B$ represent simultaneously measurable quantities. However, if there is no correlation between the relevant and reference systems, the uncertainty relation (4.44) is satisfied. Let us suppose that there is a correlation between the relevant and reference systems and consider the state given by

$$\hat{W} = |\Psi\rangle\rangle\langle\langle\Psi| \tag{4.45a}$$

$$|\Psi\rangle\rangle = (1 - g^2)^{1/2} \sum_{n=0}^{\infty} g^n |n; A\rangle \otimes |n; B\rangle \tag{4.45b}$$

with $|g| < 1$. This state is equivalent to the two-mode squeezed-vacuum state. The uncertainty relation in this state becomes

$$\Delta r \Delta r = \frac{1 - g}{1 + g} \tag{4.46}$$

Thus we obtain the relation $0 < \Delta r \Delta r < 1$ for $0 < g < 1$.

4.5. Physical Systems

The relative-position state $|\pi_s(r, k)\rangle\rangle$ is the simultaneous eigenstate of $\hat{x}_A - \hat{x}_B$ and $\hat{p}_A + \hat{p}_B$ and the phase-space probability distribution $W(r, k)\, dr\, dk$ is the probability that $\hat{x}_A - \hat{x}_B$ and $\hat{p}_A + \hat{p}_B$ take values in $(r, r + dr)$ and $(k, k + dk)$, respectively. On the other hand, the relative-momentum state $|\tilde{\pi}_s(r, k)\rangle\rangle$ is the simultaneous eigenstate of $\hat{x}_A + \hat{x}_B$ and $\hat{p}_A - \hat{p}_B$, and the phase-space probability distribution $\tilde{W}(r, k)\, dr\, dk$ is the probability that $\hat{x}_A + \hat{x}_B$ and $\hat{p}_A - \hat{p}_B$ take values in $(r, r + dr)$ and $(k, k + dk)$, respectively. Thus the relative-position and relative-momentum states and the phase-space probability distributions are closely related to the simultaneous measurement of the conjugate variables (Arthurs and Kelly, 1965; Stenholm, 1992; Yamamoto and Haus, 1986). In this section, we will briefly consider the quantum optical systems in which the measurable quantities are $\hat{x}_A \pm \hat{x}_B$ and $\hat{p}_A \mp \hat{p}_B$ (Lai and Haus, 1989).

First consider a 50%–50% lossless beam splitter (Lai and Haus, 1989). Let $\hat{a}_{\text{in}}$ and $\hat{b}_{\text{in}}$ be the annihilation operators at the two input ports of the beam splitter. Then the annihilation operators of the two outputs of the beam splitter are given by

$$\hat{a}_{\text{out}} = \frac{1}{\sqrt{2}} (\hat{a}_{\text{in}} - \hat{b}_{\text{in}}), \qquad \hat{b}_{\text{out}} = \frac{1}{\sqrt{2}} (\hat{a}_{\text{in}} + \hat{b}_{\text{in}}) \tag{4.47}$$

Using the homodyne detections for the two output signals, we can measure the quadrature components given by

$$\hat{x} = \frac{1}{\sqrt{2}}(\hat{a}_{\text{out}} + \hat{a}_{\text{out}}^\dagger) = \frac{1}{\sqrt{2}}(\hat{x}_A - \hat{x}_B) \tag{4.48a}$$

$$\hat{p} = -\frac{i}{\sqrt{2}}(\hat{b}_{\text{out}} - \hat{b}_{\text{out}}^\dagger) = \frac{1}{\sqrt{2}}(\hat{p}_A + \hat{p}_B) \tag{4.48b}$$

where $\hat{x}_A$, $\hat{x}_B$ and $\hat{p}_A$, $\hat{p}_B$ are given by

$$\hat{x}_A = \frac{1}{\sqrt{2}}(\hat{a}_{\text{in}} + \hat{a}_{\text{in}}^\dagger), \qquad \hat{p}_A = -\frac{i}{\sqrt{2}}(\hat{a}_{\text{in}} - \hat{a}_{\text{in}}^\dagger) \tag{4.49a}$$

$$\hat{x}_B = \frac{1}{\sqrt{2}}(\hat{b}_{\text{in}} + \hat{b}_{\text{in}}^\dagger), \qquad \hat{p}_B = -\frac{i}{\sqrt{2}}(\hat{b}_{\text{in}} - \hat{b}_{\text{in}}^\dagger) \tag{4.49b}$$

Thus the measured quantities are $\hat{x}_A - \hat{x}_B$ and $\hat{p}_A + \hat{p}_B$. In this case, the relative-position state $|\pi_s(r, k)\rangle\rangle$ and the phase-space probability distribution $W(r, k)$ are suitable for describing the system.

Next consider the heterodyne detection in the Shapiro–Wagner scheme (Shapiro and Wagner, 1984; Hradil, 1992). In this case, the measured quantities are the real and imaginary parts of the operator given by

$$\hat{\xi} = \hat{a} + \hat{b}^\dagger \tag{4.50}$$

where $(\hat{a}, \hat{a}^\dagger)$ and $(\hat{b}, \hat{b}^\dagger)$ stand for the signal and image band modes in the heterodyne detection, respectively. The real and imaginary parts of the operator $\hat{\xi}$ are given by

$$\hat{x} = \frac{1}{\sqrt{2}}(\hat{\xi} + \hat{\xi}^\dagger) = \hat{x}_A + \hat{x}_B \tag{4.51a}$$

$$\hat{p} = -\frac{i}{\sqrt{2}}(\hat{\xi} - \hat{\xi}^\dagger) = \hat{p}_A - \hat{p}_B \tag{4.51b}$$

where $\hat{x}$ ($\hat{x}_B$) and $\hat{p}_A$ ($\hat{p}_B$) are the position and momentum operators (or the quadrature components) of the signal (image band) mode, respectively. Thus the measured quantities are $\hat{x}_A + \hat{x}_B$ and $\hat{p}_A - \hat{p}_B$, and we find that this system is described by the relative-momentum state $|\tilde{\pi}_s(r, k)\rangle\rangle$ and the phase-space probability distribution $\tilde{W}(r, k)$.

The relative-position and relative-momentum states and the phase-space probability distributions are also suitable for describing quantum communication (Helstrom, 1974), in which it is important to simultaneously estimate the values of the operators $\hat{x}_A$ and $\hat{p}_A$ of the signal mode that carries information. But because the position operator $\hat{x}_A$ does not commute with the momentum operator $\hat{p}_A$, we instead consider the mutually commuting variables $\hat{x}_A \pm \hat{x}_B$ and $\hat{p}_A \mp \hat{p}_B$ by introducing the fictitious mode $(\hat{x}_B, \hat{p}_B)$, and so we have to extend the Hilbert space to describe the fictitious mode. This is called the Naimark (1940) extension. The quantum estimation theory is constructed in the extended Hilbert space (Holevo, 1973). The phase-space probability distribution $W(r, k)$ or $\tilde{W}(r, k)$ is thus suitable for investigating quantum communication.

4.6. Examples of the Phase-Space Probability Distributions

This section considers the phase-space probability distribution $W(r, k)$ given by equation (4.8) for several reference system states that are typical in quantum optics.

4.6.1. The Vacuum State

When the reference system is in the vacuum state $|0; B\rangle$, the statistical operator of the reference system is given by $\hat{\rho}' = |0; B\rangle\langle B; 0|$. In this case, the relation (4.6) becomes

$$\begin{aligned}\langle B; y|\hat{\rho}'|x; B\rangle &= \frac{1}{\sqrt{\pi}}\exp\left[-\frac{1}{2}(x^2 + y^2)\right] \\ &= \langle A; x|\hat{\rho}_{\text{ref}}|y; A\rangle \end{aligned} \tag{4.52}$$

Thus, it is easy to see that the operator $\hat{\rho}_{\text{ref}}$ is equal to the statistical operator for the vacuum state of the relevant system; that is, $\hat{\rho}_{\text{ref}} = |0; A\rangle\langle A; 0|$. Then we obtain

$$\hat{D}(\mu)\hat{\rho}_{\text{ref}}\hat{D}^{\dagger}(\mu) = |\mu; A\rangle\langle A; \mu| \tag{4.53}$$

where $|\mu; A\rangle$ is the coherent state of the relevant system. Therefore the phase-space probability distribution $W(r, k)$ becomes

$$W(r, k) = \tfrac{1}{2}Q(\mu), \qquad Q(\mu) = \pi^{-1}\langle A; \mu|\hat{\rho}|\mu; A\rangle \tag{4.54}$$

where $Q(\mu)$ is the Q-function (or the Husimi function) of the relevant system (Husimi, 1940; Kano, 1965; Mehta and Sudarshan, 1965).

Further assume that the relevant system is in the coherent state $|\alpha; A\rangle$ with complex amplitude $\alpha = (q + ip)/\sqrt{2}$. Then we obtain the phase-space probability distribution

$$W(r, k) = \frac{1}{2\pi}\exp\left[-\frac{1}{2}(r - q)^2 - \frac{1}{2}(k - p)^2\right] \tag{4.55}$$

Using this result, we can calculate the average values and fluctuations of the phase-space variables r and k as

$$\langle r\rangle = q, \qquad \langle k\rangle = p, \qquad (\Delta r)^2 = (\Delta k)^2 = 1 \tag{4.56}$$

It should be noted that the expectation values and fluctuations of the position and momentum operators of the relevant system in the coherent state $|\alpha; A\rangle$ are given by

$$\langle \hat{x}_A \rangle = q, \qquad \langle \hat{p}_A \rangle = p, \qquad (\Delta \hat{x}_A)^2 = (\Delta \hat{p}_A)^2 = \tfrac{1}{2} \tag{4.57}$$

It is easily seen from equations (4.56) and (4.57) that the following relations are satisfied:

$$\langle r \rangle = \langle \hat{x}_A \rangle, \qquad \langle k \rangle = \langle \hat{p}_A \rangle, \qquad (\Delta r)^2 = (\Delta \hat{x}_A)^2 + \tfrac{1}{2}, \qquad (\Delta k)^2 = (\Delta \hat{p}_A)^2 + \tfrac{1}{2} \tag{4.58}$$

The enhancement of the fluctuations calculated by the phase-space probability distribution $W(r, k)$ is due to the vacuum fluctuation of the reference system, and this enhancement is equivalent to that caused by the simultaneous measurement of the noncommuting observables (Wódkiewicz, 1987; Arthurs and Kelly, 1965; Stenholm, 1992).

4.6.2. The Number Eigenstate

Let us consider here that the reference system is in the number eigenstate $|n; B\rangle$. Substituting $\hat{\rho}' = |n; B\rangle\langle B; n|$ into equation (4.6), we obtain

$$\begin{aligned}\langle B; y|\hat{\rho}'|x; B\rangle &= \frac{1}{n!\, 2^n \sqrt{\pi}} \exp\left[-\frac{1}{2}(x^2 + y^2)\right] H_n(x) H_n(y) \\ &= \langle A; x|\hat{\rho}_{\text{ref}}|y; A\rangle \end{aligned} \tag{4.59}$$

where $H_n(x)$ is the Hermitian polynomial of order n. It is easy to see from this equation that the operator $\hat{\rho}_{\text{ref}}$ becomes the statistical operator for the number eigenstate of the relevant system; that is, $\hat{\rho}_{\text{ref}} = |n; A\rangle\langle A; n|$. Thus, the phase-space probability distribution $W(r, k)$ is given by

$$W(r, k) = \frac{1}{2\pi} \langle A; n, \mu|\hat{\rho}|\mu, n; A\rangle \tag{4.60}$$

where $|\mu, n; A\rangle = \hat{D}(\mu)|n; A\rangle$ is the displaced number state of the relevant system (Boiteux and Levelut, 1973; Mahran and Satyanarayana, 1986; de Oliveira *et al.*, 1990).

Suppose the coherent state $|\alpha; A\rangle$ of the relevant system. In this case, the phase-space probability distribution is calculated as

$$W(r, k) = \frac{1}{2^{n+1} n!\, \pi} \left[(r - q)^2 + (k - p)^2\right]^n \exp\left[-\frac{1}{2}(r - q)^2 - \frac{1}{2}(k - p)^2\right] \tag{4.61}$$

with $\alpha = (q + ip)/\sqrt{2}$. Thus, we obtain the average values and fluctuations of the phase-space variables r and k,

$$\langle r \rangle = q, \qquad \langle k \rangle = p, \qquad (\Delta r)^2 = (\Delta k)^2 = n + 1 \tag{4.62}$$

This means that the fluctuations become greater as the photon number of the reference system increases.

4.6.3. The Thermal State

Consider the thermal state of the reference system given by

$$\hat{\rho}' = \frac{1}{1 + \bar{n}} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{1 + \bar{n}} \right)^n | n; B \rangle \langle B; n | \tag{4.63}$$

where $\bar{n}$ is the bosonic distribution function. Using the result obtained for the number eigenstate of the reference system, we find that the operator $\hat{D}(\mu)\hat{\rho}_{\mathrm{ref}}\hat{D}^{\dagger}(\mu)$ is the displaced thermal state

$$\hat{D}(\mu)\hat{\rho}_{\mathrm{ref}}\hat{D}^{\dagger}(\mu) = \frac{1}{1 + \bar{n}} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{1 + \bar{n}} \right)^n | \mu, n; A \rangle \langle A; n, \mu | \tag{4.64}$$

where $| \mu, n; A \rangle$ is the displaced number state of the relevant system. Thus, we obtain the phase-space probability distribution

$$W(r, k) = \frac{1}{2\pi(1 + \bar{n})} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{1 + \bar{n}} \right)^n \langle A; n, \mu | \hat{\rho} | \mu, n; A \rangle \tag{4.65}$$

For the coherent state of the relevant system $| \alpha; A \rangle$ with complex amplitude $\alpha = (q + ip)/\sqrt{2}$, equation (4.65) becomes

$$W(r, k) = \frac{1}{2\pi(1 + \bar{n})} \exp\left[-\frac{(r - q)^2 + (k - p)^2}{2(1 + \bar{n})} \right] \tag{4.66}$$

Using this result, we can calculate the average values and fluctuations of the phase variables r and k,

$$\langle r \rangle = q, \qquad \langle k \rangle = p, \qquad (\Delta r)^2 = (\Delta k)^2 = 1 + \bar{n} \tag{4.67}$$

It is reasonable that the fluctuations become larger as the temperature of the reference system increases.

4.6.4. The Coherent State

For the coherent state $| \beta; B \rangle$ of the reference system, by substituting $\hat{\rho}' = | \beta; B \rangle \langle B; \beta |$ into relation (4.6), we obtain

$$
\begin{aligned}
\langle A; x|\hat{\rho}_{\text{ref}}|y; A\rangle
&= \langle B; y|\hat{\rho}'|x; B\rangle \\
&= \frac{1}{\sqrt{\pi}}\exp\left[-\frac{1}{2}(x^2+y^2)+\sqrt{2}(x\beta^*+y\beta)-\frac{1}{2}(\beta^2+\beta^{*2})-|\beta|^2\right]
\end{aligned}
\tag{4.68}
$$

Thus, we find that the operator $\hat{\rho}_{\text{ref}}$ is the statistical operator for the coherent state of the relevant system; that is, $\hat{\rho}_{\text{ref}} = |\beta^*; A\rangle\langle A; \beta^*|$. Since we obtain the operator $\hat{D}(\mu)\hat{\rho}_{\text{ref}}\hat{D}^\dagger(\mu) = |\mu + \beta^*; A\rangle\langle A; \mu + \beta^*|$, the phase-space probability distribution $W(r, k)$ is given by

$$W(r, k) = \tfrac{1}{2}Q(\mu + \beta^*) \tag{4.69}$$

where $Q(\mu)$ is the Q-function of the relevant system. Thus, the phase-space probability distribution is nothing but the Q-function of the relevant system. When the relevant system is also in the coherent state $|\alpha; A\rangle$, the average values and fluctuations become

$$\langle r\rangle = q - \tilde{q}, \qquad \langle k\rangle = p + \tilde{p}, \qquad (\Delta r)^2 = (\Delta k)^2 = 1 \tag{4.70}$$

where $\alpha = (q + ip)/\sqrt{2}$ and $\beta = (\tilde{q} + i\tilde{p})/\sqrt{2}$.

4.6.5. The Squeezed-Vacuum State

Finally, consider the squeezed-vacuum state of the reference system:

$$
\begin{aligned}
\hat{\rho}' &= |\xi, 0; B\rangle\langle B; 0, \xi| \\
&= \hat{S}_B(\xi)|0; B\rangle\langle B; 0|\hat{S}_B^\dagger(\xi)
\end{aligned}
\tag{4.71}
$$

where $\hat{S}_B = \exp[\frac{1}{2}(\xi\hat{b}^{\dagger 2} - \xi^*\hat{b}^2)]$ is the squeezing operator of the reference system. In the same way as for the coherent state of the reference system, we can obtain the following phase-space probability distribution $W(r, k)$:

$$W(r, k) = \frac{1}{2\pi}\langle A; \mu, \xi^*|\hat{\rho}|\xi^*, \mu; A\rangle \tag{4.72}$$

where $|\xi, \mu; A\rangle = \hat{D}(\mu)\hat{S}(\xi)|0; A\rangle$ is the squeezed-coherent state of the relevant system (Gardiner, 1991; Carmichael, 1993; Walls and Milburn, 1994; Yuen, 1976; Caves and Schumaker, 1985) and $\hat{S}(\xi) = \exp[\frac{1}{2}(\xi\hat{a}^{\dagger 2} - \xi^*\hat{a}^2)]$ is the squeezing operator of the relevant system. When the squeezing parameter is real ($\xi = \gamma$) and the relation (4.21c) is used, the phase-space probability distribution can be expressed as

$$
\begin{aligned}
W(r, k) = \frac{1}{\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp \\
\times \exp[-e^{-2\gamma}(x - r)^2 - e^{2\gamma}(p - k)^2]W_{\text{W}}(x, p)
\end{aligned}
\tag{4.73}
$$

where $W_W(x, p)$ is the Wigner function of the relevant system. It should be noted that this relation is equivalent to the generalized antinormally ordered distribution function considered in Lee (1995).

When the relevant system is in the coherent state $|\alpha; A\rangle$ with $\alpha = (q + ip)/\sqrt{2}$, the phase-space probability distribution (4.72) becomes

$$W(r, k) = \frac{1}{\pi(e^{\gamma} + e^{-\gamma})} \exp\left[-\frac{(r-q)^2}{e^{2\gamma}+1} - \frac{(k-p)^2}{1+e^{-2\gamma}} \right] \tag{4.74}$$

where we have assumed, for simplicity, that the squeezing parameter is real ($\xi = \gamma$). Thus, we obtain the average values and fluctuations of the phase space variables r and k,

$$\langle r\rangle = q, \qquad \langle k\rangle = p, \qquad (\Delta r)^2 = \tfrac{1}{2}(1 + e^{2\gamma}), \qquad (\Delta k)^2 = \tfrac{1}{2}(1 + e^{-2\gamma}) \tag{4.75}$$

The uncertainty product becomes $\Delta r \Delta k = \cosh \gamma$. For the strong squeezing limit ($|\gamma| >> 1$), we obtain

$$(\Delta r)^2 \approx (\Delta \hat{x}_A)^2 = \tfrac{1}{2} \qquad \text{for} \quad \gamma \to -\infty \tag{4.76a}$$

$$(\Delta k)^2 \approx (\Delta \hat{p}_A)^2 = \tfrac{1}{2} \qquad \text{for} \quad \gamma \to \infty \tag{4.76b}$$

Thus, if we use the squeezed-vacuum state of the reference system with an extremely large squeezing parameter, we can prevent additional noise from being introduced in the one quadrature component.

5. SUMMARY

This paper has presented the phase-space representation of quantum systems in terms of the relative-position and relative-momentum states $|\pi_s(r, k)\rangle\rangle$ and $|\tilde{\pi}_s(r, k)\rangle\rangle$ and has introduced the phase-space probability distributions $W(r, k)$ and $\tilde{W}(r, k)$. The relative-position and relative-momentum states are constructed in the extended Hilbert space consisting of those of the relevant and reference systems. The reference system can in some case be interpreted as a measurement apparatus. Using the relative-position or relative-momentum state, we have obtained the phase-space representations of the position, momentum, annihilation, and creation operators. The phase-space representation of the relevant system is obtained by projecting the extended Hilbert space into the appropriate subspace. The transformation between the phase space and position (or momentum) space has also been obtained, and the properties of a free particle and harmonic oscillator have been investigated as examples of the phase-space representation.

The phase-space probability distribution in the relative-state formulation is expressed as a convolution of the quasiprobability distributions of the relevant and reference systems, and this phase-space probability distribution is found to be closely related to the operational probability distribution when the reference system can be considered as a measurement apparatus. Note that the phase-space probability distribution is positive-definite. The characteristic function calculated by the phase-space probability distribution is expressed as a product of the characteristic functions calculated by the quasiprobability distributions of the relevant and reference systems. The uncertainty relation in the phase space, which is equivalent to the operational uncertainty relation, has also been obtained. Furthermore, the marginal distributions derived from the phase-space probability distribution have been investigated.

In this paper, the relative-state formulation has been used to construct the phase-space representation of a quantum system. It is shown that this formulation is suitable for describing relaxation processes (Ban, 1991b,c) and also applicable to thermo field dynamics (Umezawa *et al.*, 1982; Umezawa, 1993; Kowalsky *et al.*, 1988; Ezawa *et al.*, 1991), which is a real-time quantum field theory with thermal degrees of freedom. It has recently been shown that the relative-state formulation in thermo field dynamics makes it possible to construct a quantum phase operator in the most natural way (Ban, 1994b).

APPENDIX. FOCK REPRESENTATION OF THE RELATIVE-POSITION AND RELATIVE-MOMENTUM STATES

This appendix first derives the Fock representation of the relative-position state $|\pi_s(r, k)\rangle\rangle$. When the position eigenstate $|x; A\rangle$ is expanded in terms of the number eigenstate $|n; A\rangle$ such that $\hat{a}^\dagger\hat{a}|n; A\rangle = n|n; A\rangle$, and we use the expressions

$$\langle A; x|n; A\rangle = \frac{1}{(\pi^{1/2}2^n n!)^{1/2}} \exp\left(-\frac{1}{2}x^2\right) H_n(x) \tag{A1a}$$

$$|n; A\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0; A\rangle \tag{A1b}$$

where $H_n(x)$ is a Hermite polynomial of order n, then we can obtain

$$\begin{aligned} |x; A\rangle &= \sum_{n=0}^{\infty} \langle A; n|x; A\rangle|n; A\rangle \\ &= \pi^{-1/4}e^{-x^2/2} \sum_{n=0}^{\infty} \frac{1}{n!}(\hat{a}^\dagger/\sqrt{2})^n H_n(x)|0; A\rangle \\ &= \pi^{-1/4} \exp\left[-\frac{1}{2}x^2 + \sqrt{2}x\hat{a}^\dagger - \frac{1}{2}(\hat{a}^\dagger)^2\right]|0; A\rangle \end{aligned} \tag{A2}$$

In the last equality of this equation, we have used the relation

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(x) = \exp(-z^2 + 2xz) \tag{A3}$$

Similarly, we can get for the reference system

$$|y; B\rangle = \pi^{-1/4} \exp[-\tfrac{1}{2}y^2 + \sqrt{2}y\hat{b}^\dagger - \tfrac{1}{2}(\hat{b}^\dagger)^2]|0; B\rangle \tag{A4}$$

Substituting equations (A2) and (A4) into equation (2.8) and performing the Gaussian integral, we obtain the Fock representation of the relative-position state:

$$\begin{aligned} |\pi_s(r, k)\rangle\rangle &= \frac{1}{\sqrt{2\pi}} e^{i(1-s)kr/2} \int_{-\infty}^{\infty} dx\, |x + r; A\rangle \otimes |x; B\rangle e^{ikx} \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}|\mu|^2 + \mu\hat{a}^\dagger - \mu^*\hat{b}^\dagger + \hat{a}^\dagger\hat{b}^\dagger - \frac{1}{2}iskr\right) \\ &\quad \times |0; A\rangle \otimes |0; B\rangle \end{aligned} \tag{A5}$$

where the complex parameter is given by $\mu = (r + ik)/\sqrt{2}$.

Next consider the Fock representation of the relative-momentum state $|\hat{p}i_s(r, k)\rangle\rangle$. Since $\langle A; p|n; A\rangle$ is the Fourier transform of $\langle A; x|n; A\rangle$, we obtain

$$\begin{aligned} |p; A\rangle &= \sum_{n=0}^{\infty} \langle A; n|p; A\rangle|n; A\rangle \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{ipx} \sum_{n=0}^{\infty} \langle A; n|x; A\rangle|n; A\rangle \\ &= \pi^{-1/4} \exp\left[-\frac{1}{2}p^2 + i\sqrt{2}p\hat{a}^\dagger + \frac{1}{2}(\hat{a}^\dagger)^2\right]|0; A\rangle \end{aligned} \tag{A6}$$

Similarly, we can get for the reference system

$$|p; B\rangle = \pi^{-1/4} \exp[-\tfrac{1}{2}p^2 + i\sqrt{2}p\hat{b}^\dagger + \tfrac{1}{2}(\hat{b}^\dagger)^2]|0; B\rangle \tag{A7}$$

Thus the following Fock representation (2.24) is derived by substituting equations (A6) and (A7) into equation (2.19) and by performing the Gaussian integral:

$$\begin{aligned} |\tilde{\pi}_s(r, k)\rangle\rangle &= \frac{1}{\sqrt{2\pi}} e^{-i(1-s)kr/2} \int_{-\infty}^{\infty} dp\, |p + k; A\rangle \otimes |p; B\rangle e^{-ipr} \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}|\mu|^2 + \mu\hat{a}^\dagger + \mu^*\hat{b}^\dagger - \hat{a}^\dagger\hat{b}^\dagger + \frac{1}{2}iskr\right) \\ &\quad \times |0; A\rangle \otimes |0; B\rangle \end{aligned} \tag{A8}$$

It should be noted here that by substituting $s = 0$ into equations (A5) and (A8), the Fock space representations (A5) and (A8) become equivalent to those obtained in connection with the EPR problem (Fan Hong-yi and Klauder, 1994).

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